Asymptotic Statistics of Poincaré Recurrences in Hamiltonian Systems with Divided Phase Space

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By different methods we show that for dynamical chaos in the standard map with critical golden curve, the Poincaré recurrences $P(\tau)$ and correlations $C(\tau)$ decay asymptotically in time as $P \propto C/\tau \propto 1/\tau^3$. It is also explained why this asymptotic behavior starts only at very large times. We argue that the same exponent $p = 3$ should be also valid for a general chaos border. [S0031-9007(98)08272-6]

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During the last two decades the local structure of the phase space of chaotic Hamiltonian systems and area-preserving maps has been studied in great detail [1–5]. These studies allow one to understand the universal scaling properties in the vicinity of critical invariant curves where coexistence of chaos and integrability exists on smaller and smaller scales in the phase space. The most studied case is the critical golden curve with the rotation number $r_g = [111\ldots] = (\sqrt{5} - 1)/2$ for which the scaling exponents were found with high precision and it was shown that the phase space structure is self-similar and universal [2]. The most studied map with mixed integrable and chaotic components is the standard map [6] where the golden curve is critical at the chaos parameter $K = K_g = 0.971\,635\,406\,31$ [2]. It is believed that for $K > K_g$ all invariant Kolmogorov-Arnold-Moser (KAM) curves are destroyed [2].

While the local structure of divided phase space is now well understood, the statistical properties of dynamics still remain unclear in spite of the simplicity of these systems. Among the most important statistical characteristics are the correlation function decay in time $C(\tau)$ and the statistics of Poincaré recurrences $P(\tau)$. The latter is defined as $P(\tau) = N_\tau/N$, where N_τ is the number of recurrences in a given region [7] with the recurrence time $t > \tau$ and N is the total number of recurrences. According to the Poincaré theorem an orbit always returns sufficiently close to its initial position; however, the statistics of these recurrences depends on the system dynamics and is different for integrable and chaotic motion. In the case of strong chaos without any stability islands (e.g., the Arnold cat map [8]) the probability $P(\tau)$ decays exponentially with τ. This case is similar to the coin flipping where the probability to stay heads for more than τ flips also decays exponentially. However, the situation turns out to be different for a more general case of dynamics in a chaotic component of an area-preserving map with divided phase space. The first studies of $P(\tau)$ for such a case in a separatrix map showed that at a large time the recurrences decay as a power law $P(\tau) \propto 1/\tau^p$ with the exponent $p \approx 1.5$ [9]. Investigations of other different maps also indicated approximately the same value of p [10,11] even if it was remarked that p can vary from map to map, and that the decay of $P(\tau)$ can even oscillate with $\ln \tau$ [9–12]. This result is of general importance and moreover it determines the correlation function decay $C(\tau)$ via the relation $C(\tau) \propto \tau P(\tau)$ [9–12]. The statistics of $P(\tau)$ is also well suited for numerical simulations due to the natural property $P(\tau) > 0$ and statistical stability.

Such a slow decay of Poincaré recurrences was attributed to the sticking of a trajectory near a critical KAM curve which restricts the region of chaos [9–14]. Indeed, when approaching the critical curve with the border rotation number r_b, the local diffusion rate D_n goes to zero as $D_n \sim |r_b - r_n|^{\alpha/2} \sim 1/q_n^\alpha$ with $\alpha = 5$ [13], where $r_n = p_n/q_n$ are the convergents for r_b determined by a continued fraction expansion. The theoretical value $\alpha = 5$ was derived from a resonant theory of critical invariant curves [13,14] and was confirmed by numerical measurements of the local diffusion rate in the vicinity of the critical golden curve in the standard map [15]. Such a decrease of the diffusion rate near the chaos border would give the exponent $p = 3$ if everything was determined by the local properties of principal resonances p_n/q_n given by the convergents of r_b [12–14,16]. However, the value $p = 3$ is strongly different from the numerically found $p \approx 1.5$. Moreover, the special simulations for the standard and separatrix maps with the border rotation number $r_b = r_g$ give a different behavior of $P(\tau)$ and different p [9,12] in spite of the fact that the local structure of the golden critical curve is universal. Different attempts have been made to resolve this difficulty. In [17] the authors argued that a contribution from nonprincipal resonances can reduce the exponent down to $p = 2$. Other arguments based on the disconnection of principal resonance scales were proposed in [12], while Murray discussed the possibility that larger times are required to see $p = 3$ decay [18]. During this time different Hamiltonian systems were studied where the values of $p \approx 1$–2.5 have been found [11,19–22].

The analysis of Poincaré recurrences is interesting not only by itself but also because they are directly related to the correlation function of dynamical variables [10–17]:

$$C(\tau) \sim \tau P(\tau)/\langle\tau\rangle \sim 1/\tau^{p-1}, \qquad (1)$$

where $\langle\tau\rangle$ is the average recurrence time. This relation can be understood as follows. By definition $P(\tau) = N_\tau/N$; therefore, for the total time $T = \langle\tau\rangle N$ on which one investigates the recurrences of a trajectory, we have $P(\tau) = \langle\tau\rangle N_\tau/T \sim \langle\tau\rangle\mu(\tau)/\tau$. Here, due to the ergodicity of motion the measure of the sticking region $\mu(\tau) \sim T_\tau/T$ is proportional to the ratio of time the trajectory spends in the region ($T_\tau \sim \tau N_\tau$) to the total time T. Inside this region the dynamical variables are correlated so that $C(\tau) \sim \mu(\tau)$ [9,11–13]. Since the correlations are directly related to a diffusion rate ($D_c \sim \int C\, d\tau$) the exponent $p < 2$ can lead to a superdiffusive dynamics [11–14]. For the standard map such a behavior was indeed observed in [11,12,23]. All this shows that the asymptotic decay of Poincaré recurrences is a cornerstone statistical problem of two-dimensional maps.

To understand the asymptotic properties of $P(\tau)$ we used for the first time a new approach based on a direct computation of *exit* times from a vicinity of the critical golden curve in the standard map

$$\bar{y} = y - K/(2\pi)\sin(2\pi x), \qquad \bar{x} = x + \bar{y} \mod 1 \qquad (2)$$

with parameter $K = K_g$. The properties of this curve had been studied in great detail [2]. In particular, the positions of unstable fixed points of resonances p_n/q_n are known with high precision [2]. To determine the exit time τ_n from the scale q_n we placed 100 orbits in a very close vicinity of an unstable fixed point and computed the average exit time. For each orbit the exit time is determined as a time after which the orbit crosses the exit line. The exit line was fixed as $y = 1$ for the orbits from the side of the main resonance $q = 1$ or as $y = 0.5 + a\sin(2\pi x)$ for the orbits from the other side of the critical curve with $q = 2$. In the latter case the exit line was drawn in such a way to cross the two unstable points of resonance $q = 2$ ($a = 0.0773\ldots$). This allowed us to take into account the deformation of the $q = 2$ resonances. The average exit time τ_n from a given scale q_n is related to the distance of this resonance from the curve and is proportional to this distance (measure of chaos) $\mu_n = |r_g - r_n| \approx 1/\sqrt{5}\,q_n^2$ squared divided by the local diffusion rate D_n: $\tau_n \sim \mu_n^2/D_n \sim q_n$. This gives $\mu \sim C \sim 1/\tau^2$ and $p = 3$. The numerical data for dependence of μ_n on τ_n is shown in Fig. 1. From both sides of the r_g curve the exit times converge to the asymptotic dependence

$$\mu_n = (\tau_g/\tau_n)^2/\sqrt{5}, \qquad \tau_n = \tau_g q_n, \qquad \tau_g = 2.11 \times 10^5. \qquad (3)$$

This dependence corresponds to the scaling near the critical curve [2,13]. Indeed, the local diffusion rate in y on the scale q_n is $D_n \approx AD_0/q_n^5$, where $D_0 = K^2/8\pi^2$ is the quasilinear diffusion rate [8] and $A \approx 0.0066$ is a numerical constant which is quite small due to a slow diffusion inside the separatrix layer [15]. As a result the sum of transition times between the two scales from r_n to r_{n-2} is approximately equal to the total exit time: $\tau_n \approx \sum_n |r_n - r_{n-2}|^2/D_n \approx 1.4 \times 10^5 q_n$. This estimate gives the value of τ_g close to (3) and allows us to understand the physical origin of its large numerical value. It is interesting to note that the data in Fig. 1 show that the convergence of τ_n to

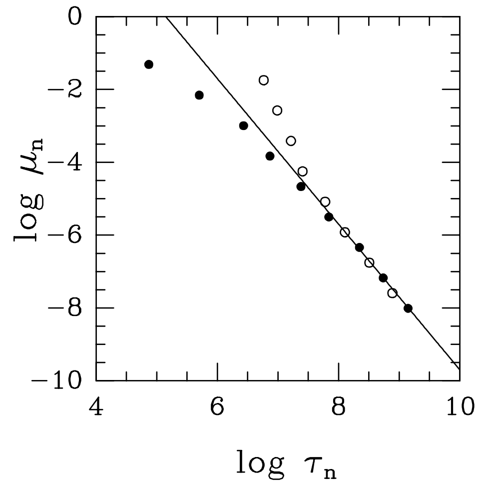

FIG. 1. Dependence of exit time τ_n from the scale r_n on a distance (chaos measure) from the critical golden curve $\mu_n = |r_g - r_n|$ for $q_n = 3, 8, \ldots, 6765$ (black circles) and $q_n = 5, 13, \ldots, 4181$ (open circles). The straight line shows asymptotic behavior (3). Error bars are less than the symbol size. Logarithms are decimal in Figs. 1–3.

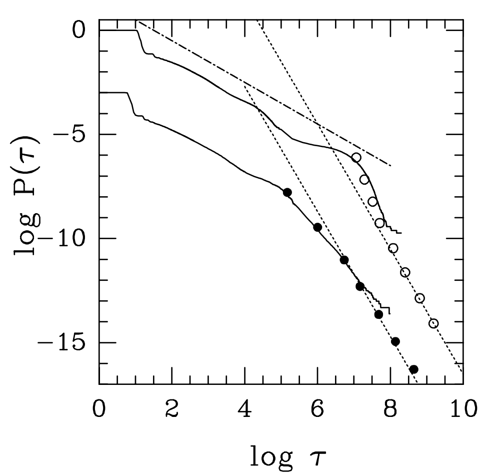

FIG. 2. Poincaré recurrences in the standard map at $K = K_g$ from the side of resonance $q = 2$ (upper full curve) and $q = 1$ (lower full curve, shifted down by 3 for clarity). Open and full circles show the values of $P(\tau)$ recalculated from the data of Fig. 1 (see text). The dashed straight lines mark the asymptotic decay with theoretical exponent $p = 3$; slope $p = 1$ is shown by the dash-dotted line. Data for $P(\tau)$ are obtained from ten orbits of length 10^{11}.

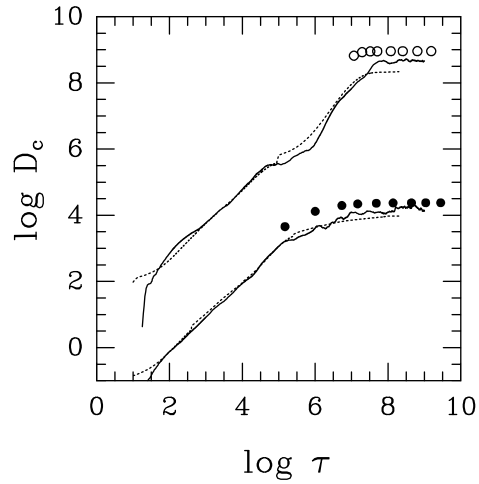

FIG. 3. Dependence of diffusion rate D_c on time (full curves) compared with its values computed from the Poincaré recurrences of Fig. 2 (dashes curves) and exit times of Fig. 1 (open and full circles) according to relation (1) (see text). Lower curves and circles are for $q = 1$ side, while the upper ones are for $q = 2$ (shifted up by 4 for clarity). For clarity, all circles are shifted up by 0.3 from their optimal positions given by coefficients $\tilde{G}_q$ (see text).

its asymptotic value can be satisfactorily described as $|\tau_n/q_n\tau_g - 1| \propto 1/q_n$. This indicates a certain similarity between the ratio $\tau_n/q_n\tau_g$ and the residue R_n for periodic orbits which converges to its critical value in a similar way [2]. The physical reason of this similarity is the following: R_n is related to the orbit stability and the larger it is, the more rapidly the orbit escapes from the scale q_n. Because of that for odd $n(q_n = 1, 3, 8, \ldots)$ the time τ_n is smaller than the asymptotic expression (3) ($R_n > R_{\mathrm{cr}} = 0.250\ldots$ [2]) while for even n ($q_n = 2, 5, 13, \ldots$) it is larger than (3) ($R_n < R_{\mathrm{cr}}$ [2]). Because of universality of the critical golden curve structure it is natural to expect that the relation (3) and the time τ_g are universal for all area-preserving maps as well as R_{cr} (note that q_n is the period of unstable periodic orbit on the scale n).

The relation (3) determines the measure of chaotic region $\mu \sim \mu_n$ at which a trajectory is stuck for a time $\tau \sim \tau_n$. Then, according to (1) the exponent of Poincaré recurrences is $p = 3$ and correlations $C(\tau) \sim \mu$ decay as inverse square of time. However, this asymptotic decay starts in fact only after a very long time $\tau > 10^6$ due to the large value of τ_g. This strong delay of asymptotic behavior is responsible for the nonuniversal decay observed for $P(\tau)$ in [11,12] at $K = K_g$. Indeed, for $\tau < \tau_g$ a trajectory does not feel the border and μ remains approximately constant giving $p = 1$ that had been seen in [11,12] (see Fig. 2). However, to observe the theoretical exponent $p = 3$ one should go to longer times. To check these theoretical expectations we made extensive numerical simulations of $P(\tau)$ at $K = K_g$ with recurrences on the exit lines defined above on both sides of the critical r_g curve. We note that $P(\tau)$ defined in this way is proportional to the probability to remain (survive) up till time τ in the region bounded by the exit line [$P(\tau) \propto d\mu/d\tau$]. The results are presented in Fig. 2 and show a clear change in decay of $P(\tau)$ for $\tau > 10^5$ (side $q = 1$) and $\tau > 10^7$ (side $q = 2$). To check the relation (1) we computed $P(\tau)$ from the data of Fig. 1 taking the recurrence time $\tau = 2\tau_n$ and $P(\tau) = B_q\langle\tau\rangle_q\mu_n/\tau$, where $q = 1, 2$ mark the side of critical curve. With the average recurrence time $\langle\tau\rangle_1 \approx 24.5$ (or $\langle\tau\rangle_2 \approx 61$) and an arbitrary constant $B_1 = 2.0$ (or $B_2 = 8.2$) the data from Fig. 1 describe the variation of $P(\tau)$ in the interval of 6 (4) orders of magnitude. This gives additional support for the theoretical exponent $p = 3$.

Another check of the relation (1) was done by computing the diffusion rate in phase $\bar{z} = z + (\bar{y} + y - 2z_q)/2$ with $z_q = 0(q = 1)$ and $z_q = 1/2(q = 2)$. A similar approach was used in [23]. The diffusion rate is $D_c = (\Delta z)^2/\Delta t$ and its dependence on time is determined by the decay of the correlation function of $y(t)$. According to (1) we have $D_c(\tau) = D_{cq}G_q\int\tau[P(\tau)/\langle\tau\rangle]\,d\tau = D_{cq}\tilde{G}_q\int C(\tau)\,d\tau$, where $D_{cq} = |r_g - r_q|^2/3 = 0.049(0.0046)$ is the quasilinear diffusion rate [8] for $q = 1$ (2) side and $G_q, \tilde{G}_q$ are some constants. The correlation $C(\tau)$ was computed from the linear interpolation of data in Fig. 1. In addition we took that $C(2\tau_n) = \mu_n/\mu_{n1}$, where μ_{n1} is the value of μ_n at the first scale on each side of the critical curve with $q = 3, 5$. In this way $C(\tau)$ remains constant up to the first exit time τ_{n1} $[C(0) = C(\tau_{n1}) = 1]$ that corresponds to the fact that $P(\tau) \sim 1/\tau$ for $\tau < \tau_{n1} \sim \tau_g$. For $p = 3$ the rate D_c should be finite. The value of D_c was computed for 100 orbits initially located near unstable fixed point of period $q = 1, 2$. The diffusion rate dependence on τ, and comparison with its computation from $P(\tau)$ and exit times $\tau_n = \tau/2$ via the above integral relation, are shown in Fig. 3. Both methods give a good agreement with $D_c(\tau)$, especially in the case of $P(\tau)$, confirming (1). The constants are $G_1 \approx G_2 \approx 2$, $\tilde{G}_1 \approx 0.3, \tilde{G}_2 \approx 0.6$. According to the above values of coefficients B_q the ratio $\tilde{G}_q/G_q$ should be approximately 2 times smaller. This may be due to the approximate scheme used to relate $C(\tau)$ with $\mu(\tau)$. For $\tau > 10^7$ the asymptotic value $p = 3$ leads to a saturation of D_c growth in time. Even if the asymptotic diffusion rate is constant $D_c = D_c(\infty)$ the distribution function is non-Gaussian since the higher moments diverge. For smaller τ the diffusion rate D_c grows approximately linearly that corresponds to an intermediate value of $p \approx 1$. This intermediate slow decay is responsible for the enormously large ratio of the asymptotic diffusion rate to its quasilinear value: $D_c(\infty)/D_{cq} \approx 3 \times 10^5(q = 1)$; $10^7(q = 2)$.

The ensemble of data in Figs. 1–3 shows that the asymptotic decay of Poincaré recurrences and correlations is determined by the universal structure in the vicinity of the critical golden curve, and the contribution of boundaries of other internal islands of stability is not significant at variance with [17]. The hypothesis of dynamical disconnection of scales [12] is also ruled out. Our results are in agreement with previous numerical observations indicating that long recurrences are related to orbits being very close to the r_g curve [11,12]. It is interesting to ask the question how the value of the exponent $p = 3$ would be modified for the case of a main border curve $r_b = [a_1, a_2, \ldots, a_i, \ldots]$ with nongolden continued fraction expansion. The numerical data for border invariant curves obtained in [5] show that the elements a_i mainly take the values 1, 2, 3, 4 and the probability to find $a_i > 4$ is rather small. For such bounded a_i values the general resonant approach developed in [12–14] still shows that the diffusion rate near the border scales as $D_n \propto 1/q_n^5$, and, therefore, the exit time will scale as $\tau_n \sim q_n$ giving the exponent $p = 3$. Because of the similarity between τ_n/q_n and the residue R_n discussed above we can expect that, for a typical r_b, the ratio τ_n/q_n will not converge to a constant as it was for $r_b = r_g$ but will oscillate in a bounded interval similar to the oscillations of R_n (see, e.g., [12]). Because of the above reasons we can expect that even in the general case of nongolden border invariant curves the average asymptotic universal exponent is $p = 3$.

If the exponent p is 3, then it is natural to ask why previous computations of different groups were giving $p \approx 1.5$. Our explanation is based on the following arguments. First, even for the critical golden curve the asymptotic regime starts after a very long time which is determined by the first resonance scales. The first scales are not universal; this explains why p was varying from map to map. If the border curve is nongolden, then the ratio τ_n/q_n should oscillate with n and the asymptotic regime will appear even later than for $r_b = r_g$. Also on the first scales a given map can be often locally approximated by the standard map with $K \approx K_{\text{cr}} + (r - r_b)df(r)/dr$, where f is some smooth function of the rotation number [6]. A typical example is the separatrix map [6,11,12]. In this case at a given r_n the local order parameter is supercritical with $K - K_{\text{cr}} \propto |r_n - r_b| \propto 1/q_n^2$. This scaling is different from the asymptotic one with $K - K_{\text{cr}} \propto 1/q_n \propto |r_n - r_b|^{1/2}$ [2,12,18] and can give a very long exit time for first scales. Indeed, in the standard map with $K > K_g$ the transition time from $y = 0$ to $y = 1$ is proportional to $1/|K - K_g|^3$ [4,6] thus giving an exit time $\tau_n \sim /|K - K_g|^3 \sim q_n^6$. According to (1) this will give $p = 4/3$ which is not far from the average $p \approx 1.5$. In addition, when close to the critical curve, as in the standard map with $K = K_g$, one should still wait a long time τ_g to reach the asymptotic exponent $p = 3$.

In conclusion, we have shown that in the case of the critical golden curve the asymptotic exponent for the decay of Poincaré recurrences is $p = 3$. This implies that the correlation integral converges and the diffusion rate produced by such dynamical chaos is finite. However, the higher moments of distribution will diverge. We argue that the asymptotic exponent should remain the same also in the case of a typical border invariant curve.

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