

Quantum Chaos and Quantum Computers

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Abstract

The standard generic quantum computer model is studied analytically and numerically and the border for emergence of quantum chaos, induced by imperfections and residual inter-qubit couplings, is determined. This phenomenon appears in an isolated quantum computer without any external decoherence. The onset of quantum chaos leads to quantum computer hardware melting, strong quantum entropy growth and destruction of computer operability. The time scales for development of quantum chaos and ergodicity are determined. In spite the fact that this phenomenon is rather dangerous for quantum computing it is shown that the quantum chaos border for inter-qubit coupling is exponentially larger than the energy level spacing between quantum computer eigenstates and drops only linearly with the number of qubits n . As a result the ideal multi-qubit structure of the computer remains rather robust against imperfections. This opens a broad parameter region for a possible realization of quantum computer. The obtained results are related to the recent studies of quantum chaos in such many-body systems as nuclei, complex atoms and molecules, finite Fermi systems and quantum spin glass shards which are also reviewed in the paper.

1. Introduction

On the border between two Millennia it is natural to ask the question, *what will be the origin of future human power?* Even thirty or twenty years ago the standard answer would be: nuclear. But now in the view of the amazing computer development all over the world it becomes clear that the future power will be related to ability to count as fast as possible. In Millennium I this ability was basically comparable with finger counting, while at the end of Millennium II it made an enormous jump with computer creation which led to a qualitative change in human society. During last two decades the power of modern computers demonstrated a constant impressive growth due to technological progress and creation of chips of smaller and smaller size. In a near future this size should reach a scale at which the quantum nature of physical laws will become dominant. As a result, we unavoidably come to the problem of creating a quantum computer. Such a computer should be essentially based on quantum mechanics and operate with unitary transformations and quantum logic. The unitary nature of transformations allows to exclude energy dissipation that should play an important role on small scales. At the same time, as stressed by Feynman [1], the classical computer has enormous difficulty in simulation of many-body quantum systems due to exponential growth of the Hilbert space with the number of particles and hence, of the computational efforts. Due to that it can be expected that a computer composed of quantum elements will be much more efficient for solution of quantum, and maybe other, problems. At present the quantum computer is viewed as a system of n coupled qubits, being two-level quantum systems or one-half

spins (see recent review [2] and references there in). The computer operation is based on a controlled series of two-qubit coupling switches, on and off which together with one-qubit rotations allow to realize any unitary operation in the Hilbert space of size $N_H = 2^n$ [3,4]. In this respect the inter-qubit coupling becomes an unavoidable property of quantum computer.

Recently a great increase of interest in quantum computing has been generated by the work of Shor [5] who constructed a quantum algorithm which performs large number factorization into primes exponentially faster than any known classical algorithm. Also Grover showed [6] that a search of an item in a long list is done much faster by a quantum computer. The enormous gain in the computation rate is reached due to high parallelism of multi-qubit quantum evolution and quantum interference. Together with a recent theoretical development of quantum error-correcting codes [7,8] these exciting results stimulated various experimental proposals for realization of a quantum computer. The variety of physical systems proposed is really amazing and includes: ion traps [9], nuclear magnetic resonance systems [10], nuclear spins with interaction controlled electronically [11,12] or by laser pulses [13], quantum dots [14], Cooper pair boxes [15], optical lattices [16] and electrons floating on liquid helium [17]. At present two-qubit gates were experimentally realized with cold atoms [18], and the Grover algorithm has been performed for three qubits made from nuclear spins in a molecule [19].

Thus there are two main lines in the present day quantum computer research: construction and development of efficient quantum algorithms and search for an optimal physical system with a future experimental realization of few coupled qubits. The first line has a strong mathematical shade: indeed, it assumes that all qubits are perfectly identical and the couplings between them can be operated in a perfect way. The second line, in a large respect, is a part of experimental physics with few qubits. As a result, there is a broad open field for physical studies of a realistic quantum computer with many qubits. Indeed, in reality the qubits are never perfect, the level spacing fluctuates from one qubit to another due to different environment, there is also a residual interaction between qubits which cannot be eliminated completely, the two-qubit gate operations are also not perfect. In practice, at least around $n = 1000$ of such qubits are required to make a quantum computer more efficient than the existing computers today [2]. In addition to the above internal imperfections there is also decoherence due to coupling to the external world which produces noise and dissipation. The effects of decoherence and two-qubit gates pulse broadening were numerically tested on Shor's algorithm [20,21] and were shown to play an important role.

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At the same time the estimates show that it is possible to have physical qubits with very long relaxation time, during which many gate operations can be realized [22]. One of the most promising systems seems to be nuclear spins in two-dimensional semiconductor structures [11,12,23].

However, the absence of external decoherence does not yet mean that the computer will operate properly. Indeed, internal imperfections with inter-qubit residual couplings J can strongly modify the ideal quantum register represented by noninteracting many-body (multi-qubit) states of ideal qubits. A simple estimate [24] shows that the residual interaction J will be unavoidably much larger than the energy level spacing Δ_n between adjacent eigenstates of a realistic/generic quantum computer. Let us assume that J is relatively weak comparing to the one-qubit level spacing Δ_0 . Then all N_H eigenenergies will be located in an energy band of size $\Delta E \sim n\Delta_0$ and the average multi-qubit level spacing is $\Delta_n \approx \Delta E/N_H \sim \Delta_0 2^{-n} \ll \Delta_0$. For the experimental proposals [11,12] we have $\Delta_0 \sim 1$ K so that for $n = 1000$, when Shor's algorithm becomes useful, the multi-qubit spacing is incredibly small $\Delta_n \sim 10^3 \times 2^{-10^3} \Delta_0 \sim 10^{-298}$ K. This value will definitely be much smaller than any physical residual interaction J . For the proposal [12] with a distance between nuclear spins of $r = 200$ Å and an effective Bohr radius of $a_B = 30$ Å (Eq. (2) of [12]), the coupling between qubits (spin-spin interaction) is $J \sim \Delta_0 \sim 1$ K. By changing the electrostatic gate potential, an effective barrier between nuclei can be modified that can be viewed as a change of effective electron mass possibly up to a factor of two. Since $J \propto (r/a_B)^{5/2} \exp(-2r/a_B)/a_B$, and a_B is inversely proportional to the effective mass, this gives a minimal residual spin-spin interaction of $J \sim 10^{-5}$ K $\gg \Delta_n$. On the first glance this would lead to a natural/naive conclusion that at such residual interaction the ideal quantum computer eigenstates are strongly mixed and completely modified resulting in destruction of quantum computer hardware. In spite of this expectation it has been shown recently that the ideal qubit structure is much more robust and in reality the quantum hardware melting and quantum chaos induced by inter-qubit interaction takes place at $J > J_c \sim \Delta_0/n$ being exponentially larger than Δ_n [24]. This result for quantum chaos border in quantum computing is recently confirmed by more extended studies [25] and opens a broad regime of parameters for which realization of a quantum computer is possible. For example, at $\Delta_0 \sim 1$ K and $n = 1000$ the critical coupling $J_c \sim 1$ mK is compatible with the experimental proposal [12].

The above result is closely related to the long term research of quantum many-body systems, started by Wigner [26] interested in "the properties of the wave functions of quantum mechanical systems which are assumed to be so complicated that statistical consideration can be applied to them". As a result, the random matrix theory (RMT) has been developed to explain the generic properties of complex eigenstates and energy spectra of many-body interacting systems such as heavy nuclei, many electron atoms and molecules [27]. The success of RMT was so impressive [28] that the conditions of its applicability to many-body systems were not really realized until recently. Indeed, for example, in nuclei the density of states grows exponentially with excitation energy and at a first glance the RMT is valid as soon as the interaction matrix elements

U become comparable to the level spacing between many-body states Δ_n . However, in nature we have only two-body interaction and therefore, while the size of the Hilbert space grows exponentially with the number of particles n , the number of nonzero interaction induced matrix elements grows not faster than n^2 . To study the spectral statistics in this situation a two-body random interaction model (TBRIM) was introduced and it was shown [29,30] that in the limit of strong interaction the RMT remains valid even if the full Hamiltonian matrix is exponentially sparsed. However, much more time was needed to understand the case of relatively weak interaction and to find the critical U_c above which quantum chaos and RMT set in. Contrary to the naive expectation $U \sim \Delta_n$ assumed by many authors until recently (see e.g. [31–33]) it now became clear that the quantum chaos border is exponentially larger than Δ_n since only transitions between directly coupled states, and not the bulk density of states, are important for level mixing. As a result the interaction should be larger than the energy level spacing Δ_c between directly coupled states $U > U_c \sim \Delta_c \gg \Delta_n$ to change the level spacing statistics $P(s)$ from the Poisson distribution to the Wigner-Dyson one and to generate quantum ergodicity and chaos for eigenstates. As for my knowledge, for the first time this condition for quantum chaos in many-body systems had been formulated by Åberg who also by numerical simulations for a nucleus model had found the change in the level statistics across this border [34]. Due to that I will call this condition *Åberg criterion*.

In spite of other Åberg papers (e.g. [35]) his result was not broadly known for the community probably because his studies were mainly addressed to nuclear physicists and the numerical results were not confirmed at that time. Also in nuclei the interaction is usually quite strong and quantum chaos sets in rather quickly. In fact it was shown by numerical simulations of other models that in an isolated many-body Fermi system a sufficiently strong interaction can induce dynamical thermalization with the Fermi-Dirac distribution [31,32]. Independently, the line of research related to the problem of two interacting particles in a random potential [36] showed that a two-body interaction can lead to a number of unexpected results and that disorder/chaos can strongly enhance the interaction. The latter property had in fact been known from the spectacular Sushkov-Flambaum enhancement of weak interaction in nuclei [37]. The analytical studies of few particles with two-body random interaction U [38] showed that the mixing of levels, quantum chaos and RMT statistics appear only if $U > \Delta_2$, where $\Delta_2 \gg \Delta_3 \gg \Delta_4 \dots$ are the level spacings between 2-,3-,4-particle levels respectively. This result was generalized for many particles and the quantum chaos border $U_c \sim \Delta_c$ was proposed and confirmed by extensive numerical studies of the TBRIM model [40], independently of Åberg's papers (see also [39]). As it will be seen from the present paper the knowledge obtained for quantum many-body systems can be successfully used for such a new direction of research as quantum computing.

In this paper I review the recent developments of quantum chaos theory in many-body systems obtained in Toulouse and show their links and importance for a quantum computer which also represents a many-body system with exponentially large Hilbert space. The paper is organized

as follows. In the next Section the analytical and numerical results are presented for emergence of quantum chaos in many-body systems. In Section 3 the standard generic quantum computer (SGQC) model is described and the results of Section 2 are generalized for quantum chaos border for quantum computing. In Section 4 the time evolution in the regime of quantum chaos is studied and different time scales imposed by chaos and decoherence on quantum computing are discussed. The paper ends by the concluding remarks in the last Section.

2. Åberg criterion for emergence of quantum chaos in many-body systems

Following [38] let us consider first a case of three particles located on m one-particle orbitals with energy level spacing $\Delta \sim V/m$. For simplicity we assume the particles to be distinguishable which however is not of principal importance for $m \gg 1$. The level spacing between 2,3-particle states is $\Delta_2 \sim V/m^2$, $\Delta_3 \sim V/m^3$ respectively with $\Delta \gg \Delta_2 \gg \Delta_3$. The two-body matrix elements written in the noninteracting eigenbasis are supposed to be random with typical values $U_{12} \sim U$ and $U_{23} \sim U$ for interaction between 1st/2d and 2d/3d particles respectively. At the same time the interaction matrix element U_{13} between 1st/3d is taken to be zero for simplicity (the final result remains the same for $U_{13} \sim U$). For two interacting particles, e.g. 1st/2d, the levels become mixed by interaction and RMT spectral statistics sets in when the interaction becomes larger than the level spacing between two-particles states: $U_{12} > \Delta_2$. On the contrary for $U_{12} \ll \Delta_2$ the perturbation theory is valid and the eigenstates with interaction are determined by one noninteracting eigenstate. In the case of three particles the situation is more complicated since the three-particle states are not directly coupled by two-body interaction.

The matrix element U_3 between 3-particle levels can be found by the second order perturbation theory. In this way the matrix element between initial state $|123\rangle$ and final state $|1'2'3'\rangle$ is given by the diagram presented in Fig. 1 with intermediate state $|1'\bar{2}3\rangle$.

$$U_3 = \sum_{\bar{2}} \frac{\langle 12|U_{12}|1'\bar{2}\rangle \langle \bar{2}3|U_{23}|2'3'\rangle}{(E_1 + E_2 + E_3 - E_{1'} - E_{\bar{2}} - E_3)} \sim \frac{U^2}{\Delta}. \quad (1)$$

It is important that the summation is carried out only over single particle states $\bar{2}$ and the sum is mainly determined by a term with a minimal detuning in the denominator being of order Δ . As a result the level mixing sets in for $U_3 > \Delta_3$

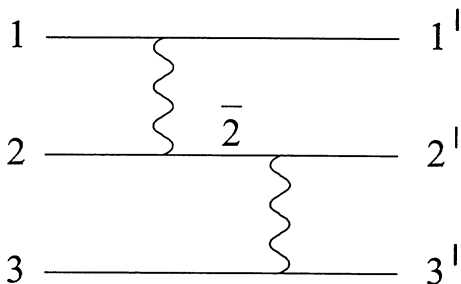


Fig. 1. Diagram for the effective 3-particle matrix element U_3 in (1), after [38].

which gives the quantum chaos border [38]:

$$U \sim U_c \sim \Delta_2 \gg \Delta_3 \quad (2)$$

This means that the 3-particle levels are mixed only when the interaction mixes two-particle levels that is the consequence of the two-body nature of interaction. In a similar way for few particles $n \sim 3$ the border $U_c \sim \Delta_2$. This conclusion was confirmed in [39].

Let us now consider a more general case of TBRIM [29–31] with n Fermi particles distributed over m energy orbitals $\varepsilon_{m'}$, $m' = 1, 2, \dots, m$. These energies are randomly and homogeneously distributed in the interval $[0, m\Delta]$ with spacing Δ . The total number of many-body states is $N = m!/(n!(m-n)!)$ and they are coupled by random two-body matrix elements with values uniformly distributed in the interval $[-U, U]$. Due to the two-body nature of interaction the number of multi-particle states coupled by interaction, or the number of direct transitions, is $K = 1 + n(m-n) + n(n-1)(m-n)(m-n-1)/4$ [31]. All these transitions occur inside a two-body energy interval $B = (2m-4)\Delta$ around the energy of the initial multi-particle state. For large m and n the number of transitions K is much smaller than the size of the matrix N but much larger than the number of different two-body matrix elements $N_2 \approx m^2/2$. The Fermi energy of the systems is $\varepsilon_F \approx n\Delta$ and the level spacing in the middle of the total energy band is exponentially small $\Delta_n \approx (m-n)n\Delta/N$.

At the middle of the spectrum the density of *directly coupled* states is $\rho_c = K/B$. According to the usual perturbation theory, these states, if they would be alone, are mixed if the coupling U is larger than their level spacing $\Delta_c = 1/\rho_c$. This is the Åberg criterion [34,35], which was also independently proposed in [40]. This relation is also in agreement with the arguments given above for few particles. As a result, the onset of quantum chaos is expected for $U > U_c$:

$$U_c = C\Delta_c = C\frac{B}{K} \approx \frac{2C}{\rho_2 n^2} \quad (3)$$

where C is a numerical constant to be found and $\rho_2 \approx N_2/B \approx m/4\Delta$ is the two-particle density at $m \gg n \gg 1$. It is important to stress that the critical coupling U_c is exponentially larger than the level spacing Δ_n between multi-particle states.

The most direct way to detect the emergence of quantum chaos is by the change in the probability distribution $P(s)$ of nearest level spacings s , where s is measured in units of average spacing. Indeed, for integrable systems the levels are noncorrelated and characterized by the Poisson distribution $P_P(s) = \exp(-s)$, while in the quantum chaos regime the statistics is close to the Wigner surmise $P_W(s) = (\pi s/2) \exp(-\pi s^2/4)$ [41]. To identify a transition from one limiting case to another it is convenient to introduce the parameter $\eta = \int_0^{s_0} (P(s) - P_W(s)) ds / \int_0^{s_0} (P_P(s) - P_W(s)) ds$, where $s_0 = 0.4729\dots$ is the intersection point of $P_P(s)$ and $P_W(s)$. In this way η varies from 1 ($P(s) = P_P(s)$) to 0 ($P(s) = P_W(s)$) and the critical value of U_c can be determined by the condition $\eta(U_c) = \eta_c = 0.3$ [40]. In fact the choice of η_c affects only the numerical constant C in (3). The chosen $\eta_c = 0.3$ is close to the value $\eta_c \approx 0.2$ for the critical statistics at the Anderson transition on a 3-dimensional disordered lattice [42]. The results of extensive numerical

studies, performed in [40], are presented in Fig. 2 and confirm the Åberg criterion (3) in a large parameter range with $C = 0.58$ [43]. Of course, the data [40] are obtained for a much larger parameter range than in [34,35]. Nevertheless, the above C value is in good agreement with a numerical factor found in [35] (see Eq. 22 there), which gives $C \approx 0.7$, if to take into account that C is defined via the same average values of square matrix element. A similar value is also found in more advanced studies for $n = 3, 4$ in the layer model approximation with the states selected in the energy interval Δ (see insert in Fig. 3) [44]. Also the above studies [34,35,40,44] show that contrary to the sharp Anderson transition [42] a smooth crossover from one statistics to another takes place at $U \approx U_c$ (see, however, [45] and references therein).

The Åberg criterion can be applied not only to excited states but also to low energy excitations near the Fermi level. Suppose we have a Fermi gas with a temperature $T \ll \varepsilon_F$. Then, according to the Fermi-Dirac distribution, the number of effectively interacting particles is $\delta n \sim Tn/\varepsilon_F$ with

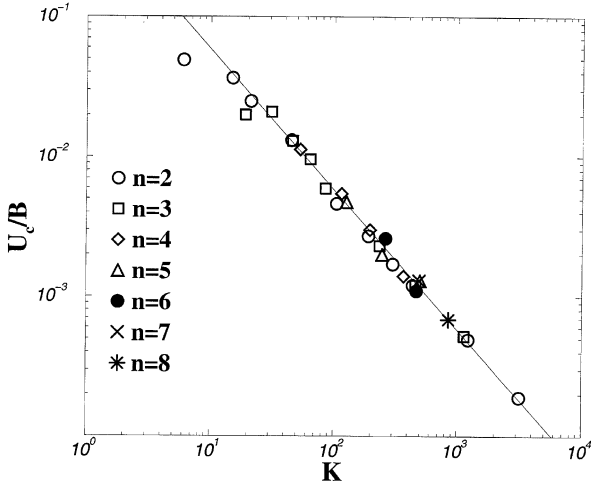


Fig. 2. Dependence of the rescaled critical interaction strength U_c/B , above which $P(s)$ becomes close to the Wigner-Dyson statistics, on the number of directly coupled states K for $4 \leq m \leq 80$ and $1/40 \leq n/m \leq 1/2$. The line shows the theory (3) with $C = 0.58$, after [40].

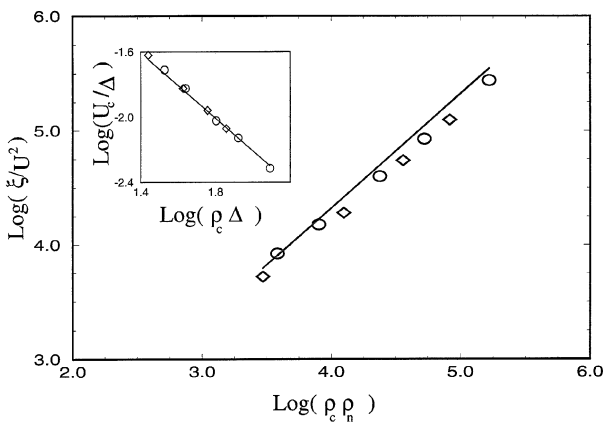


Fig. 3. Dependence of the rescaled IPR ξ/U^2 on $\rho_c \rho_n$: layer model data for $n = 3$, $\Delta = 1$ and $40 \leq m \leq 130$ (o); $n = 4$ and $30 \leq m \leq 60$ (diamonds). The straight line gives theory (5). Insert shows U_c/Δ vs. $\rho_c \Delta$ in log-log scale for the same parameters; the straight line is the fit $U_c = 0.62/\rho_c$. After [44].

the density of these two-particle states $\rho_2 \sim T/\Delta^2$ and $\rho_c \sim \rho_2 \delta n^2 \sim \Delta(T/\Delta)^3$. The total excitation energy is $\delta E \sim T \delta n \sim T^2/\Delta$. As a result the interaction induced/dynamical thermalization and the quantum chaos set in [40] only for

$$\delta E > \delta E_{ch} \approx \Delta(\Delta/U)^{2/3}; \quad T > T_{ch} \approx \Delta(\Delta/U)^{1/3}. \quad (4)$$

These relations follow also from the Åberg papers [34,35] even if they were not written directly there. The numerical constant in (4) corresponds to $\eta_c = 0.3$ [40]. Below the border (4) the eigenstates are not ergodic and the interaction is too weak to thermalize the fermions even if the multi-particle level spacing is exponentially small ($\Delta_n \propto \exp(-2.5(\delta E/\Delta)^{1/2})$ [47]). After [34,35,40] the dependence (4) was also obtained in [46].

In the quantum chaos regime $U > U_c$ the local density of states is described by the Breit-Wigner distribution with the energy width Γ given by the Fermi golden rule $\Gamma = 2\pi U^2 \rho_c/3$ [44]. The value of Γ determines the spreading width of eigenstates mixed by interaction. The number of noninteracting eigenstates contributing to a given eigenstate can be measured through the inverse participation ratio (IPR) $\xi = 1/\sum_i |a_i|^4$, where a_i are probability amplitudes in the noninteracting eigenbasis. The mixing of all levels in the interval Γ gives [44]

$$\xi \approx \Gamma \rho_n \approx 2U^2 \rho_c \rho_n. \quad (5)$$

where the numerical factor is taken in analogy with the known result for band random matrices. This analytical relation is in a good agreement with the numerical data shown in Fig. 3. It is important to note that at $U = U_c$ exponentially many states are mixed by interaction. The width Γ has also another important physical meaning: it determines the chaotic time scale $\tau_\chi \approx 1/\Gamma$ after which an initial noninteracting eigenstate disintegrates over exponentially many (ξ) eigenstates of the interacting system (here and below $\hbar = 1$).

3. Standard generic quantum computer model

In [24] the standard generic quantum computer (SGQC) model was introduced to describe a system of n qubits containing imperfections which generate a residual inter-qubit coupling and fluctuations in the energy spacings between the two states of one qubit. The Hamiltonian of this model reads:

$$H = \sum_i \Gamma_i \sigma_i^z + \sum_{i < j} J_{ij} \sigma_i^x \sigma_j^x, \quad (6)$$

where the σ_i are the Pauli matrices for the qubit i and the second sum runs over nearest-neighbor qubit pairs on a two-dimensional lattice with periodic boundary conditions applied. The energy spacing between the two states of a qubit is represented by Γ_i randomly and uniformly distributed in the interval $[\Delta_0 - \delta/2, \Delta_0 + \delta/2]$. The detuning parameter δ gives the width of the distribution near the average value Δ_0 and may vary from 0 to Δ_0 . Fluctuations in the values of Γ_i appear generally as a result of imperfections, e.g. local magnetic field and density fluctuations in the experimental proposals [11,12]. The couplings J_{ij} represent the residual static interaction between qubits which is always present

for reasons explained in the introduction. They can originate from spin-exciton exchange [11,12], Coulomb interaction [9], dipole-dipole interaction [17], etc... To catch the general features of the different proposals, J_{ij} are chosen randomly and uniformly distributed in the interval $[-J, J]$. This SGQC model describes the quantum computer hardware, while the gate operation in time should include additional time-dependent terms in the Hamiltonian (6) and will be studied separately. At $J = 0$ the noninteracting eigenstates of the SGQC model can be presented as $|\psi_i\rangle = |\alpha_1, \dots, \alpha_n\rangle$ where $\alpha_k = 0, 1$ marks the polarization of each individual qubit. These are the ideal eigenstates of a quantum computer called quantum register states. For $J \neq 0$, these states are no longer eigenstates of the Hamiltonian, and the new eigenstates are now linear combinations of different quantum register states. The term multi-qubit states is used to denote the eigenstates of the SGQC model with interaction but also for the case $J = 0$. It is important to stress that the quantum computer operates in the middle of the energy spectrum where the density of states is exponentially large and it is natural that the quantum chaos initially sets in this bulk part of the spectrum. In this respect low energy excitations are not important contrary to fermionic systems discussed above.

It is interesting to note that when one site in (6) is coupled with all other sites and $\delta = 2\Delta_0$ then the system becomes equivalent to the quantum version of the classical Sherrington-Kirpatrick spin glass model studied in [48]. For such a quantum spin glass shard in the middle of the spectrum the coupling matrix element is $U \sim J$, the spacing between directly coupled states is $\Delta_c \sim \Delta_0/n^2$, since each state is coupled to $n(n-1)/2$ states in the energy band of the order of Δ_0 . Therefore, according to (3) the quantum chaos and ergodicity emerge for $J > \Delta_0/n^2 \gg \Delta_n$ as it was shown analytically and numerically in [48].

A similar analysis can be done for the SGQC model (6). Indeed, for $\delta \ll \Delta_0$ and $J < \delta$ the total spectrum is composed of n bands with inter-band spacing $2\Delta_0$ and band width $\sqrt{n}\delta$. Within one band one quantum register state is coupled to about n states in an energy window of 2δ so that $\Delta_c \sim \delta/n$ and the quantum chaos border is given by [24]

$$J_c = C_q \delta / n \quad (7)$$

where C_q is a numerical factor. All above arguments remain valid up to $\delta = \Delta_0$ when the bands become overlapped. It is important that $J_c \gg \Delta_n$ and that J_c decreases with δ . The last property is natural since at small δ the states in one band are quasi-degenerate and it is easier to mix levels.

Direct numerical simulations for the quantum chaos border in a quantum computer are done in [24] for the SGQC model. The change in the level spacing statistics $P(s)$ in the band center with the growth of residual interaction J can be quantitatively characterized by the parameter η . To suppress fluctuations $P(s)$ is obtained by averaging over $5 \leq N_D \leq 4 \times 10^4$ random realizations of Γ_i, J_{ij} so that the total spacing statistics was $10^4 < N_S \leq 1.6 \times 10^5$. Also $P(s)$ is determined inside one of the symmetry classes of (6) with odd or even number of qubits up. The dependence of η on J is presented in Fig. 4. The variation of critical coupling with n and δ can be determined from the condition $\eta(J_c) = 0.3$. The data obtained are in a good agreement with (7) with $C_q = 3.16$ and clearly show that $J_c \gg \Delta_n$ (see inserts

in Fig. 4). According to Fig. 4 the transition is sharp in the limit of large n in contrast to the smooth crossover in the TBRIM. This difference is due to local interaction between particles in (6) contrary to the long range interaction in the TBRIM.

In the limit $\delta \ll \Delta_0$ and $J \ll \Delta_0$ the coupling between different energy bands is negligibly small. In this case to a good approximation the SGQC Hamiltonian (6) can be reduced to the renormalized Hamiltonian $H_P = \sum_{k=1}^{n+1} \hat{P}_k H \hat{P}_k$ where $\hat{P}_k$ is the projector on the k^{th} band, so that qubits are coupled only inside one band. For an even n this band is centered exactly at $E = 0$, while for odd n there are two bands centered at $E = \pm \Delta_0$, and we will use the one at $E = -\Delta_0$. Such a band corresponds to the highest density of states, and in a sense represents the quantum computer core. The dependence of critical coupling, determined via $\eta(J_c) = 0.3$, on the number of qubits n is shown in Fig. 5 being in good agreement with the theoretical quantum chaos border (7). It is important to note that at $\delta = 0$ the parameter $\eta = 0$ and the eigenstates are chaotic [25] in agreement with (7).

The drastic change in the level spacing statistics is in fact related to a qualitative change in the quantum computer eigenstate structure. For $J \ll J_c$ the eigenstates are very close to noninteracting multi-qubit states $|\psi_i\rangle$, while for $J > J_c$ each eigenstate $|\phi_m\rangle$ becomes a mixture of exponentially many $|\psi_i\rangle$. It is convenient to characterize the complexity of an eigenstate $|\phi_m\rangle$ by the quantum eigenstate entropy $S_q = -\sum_i W_{im} \log_2 W_{im}$, where W_{im} is the quantum probability to find the state $|\psi_i\rangle$ in the eigenstate $|\phi_m\rangle$ ($W_{im} = |\langle \psi_i | \phi_m \rangle|^2$). In this way $S_q = 0$ if $|\phi_m\rangle$ is represented by one $|\psi_i\rangle$, $S_q = 1$ if two $|\psi_i\rangle$ with equal probability contribute to one $|\phi_m\rangle$ and the maximal value $S_q = n$ corresponds to a mixture of all N_H states in one $|\phi_m\rangle$. A mixture of two states $|\psi_i\rangle$ is already sufficient to modify strongly the quantum register computations and it is natural to determine the critical

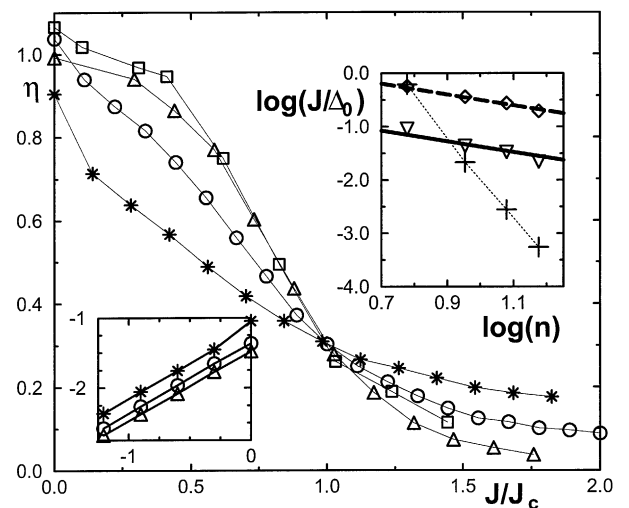


Fig. 4. Dependence of η on the rescaled coupling strength J/J_c for the states in the middle of the energy band for $n = 6$ (*), 9 (o), 12 (triangles), 15 (squares); $\delta = \Delta_0$. The upper insert shows $\log(J_c/\Delta_0)$ (diamonds) and $\log(J_{cs}/\Delta_0)$ (triangles) versus $\log(n)$; the variation of the scaled multi-qubit spacing Δ_n/Δ_0 with $\log(n)$ is shown for comparison (+). Dashed line gives the theoretical formula (7) with $C_q = 3.16$; the solid line is $J_{cs} = 0.41\Delta_0/n$. The lower insert shows $\log(J_{cs}/\Delta_0)$ versus $\log(\delta/\Delta_0)$ for $n = 6$ (*), 9 (o), 12 (triangles); straight lines have slope 1. After [24].

coupling J_{cs} by the condition $S_q(J_{cs}) = 1$. The results for the J_{cs} dependence on n and δ are shown in Figs. 4,5. They clearly show that $J_{cs} \approx 0.13J_c \approx 0.4\delta/n \gg \Delta_n$.

In the quantum chaos regime at $J > J_c$ one eigenstate is composed of ξ states $|\psi_i\rangle$ mixed in the Breit-Wigner width $\Gamma \sim J^2/\Delta_c \sim J^2n/\delta$. As for the TBRIM the IPR is exponentially large and is given by (5) [49]. For $J > \delta$ the interaction becomes too strong and $\Gamma \sim J\sqrt{n}$ [25]. The pictorial image of the quantum computer melting is shown in Fig. 6. For $J > J_c$ the eigenstates become very complex

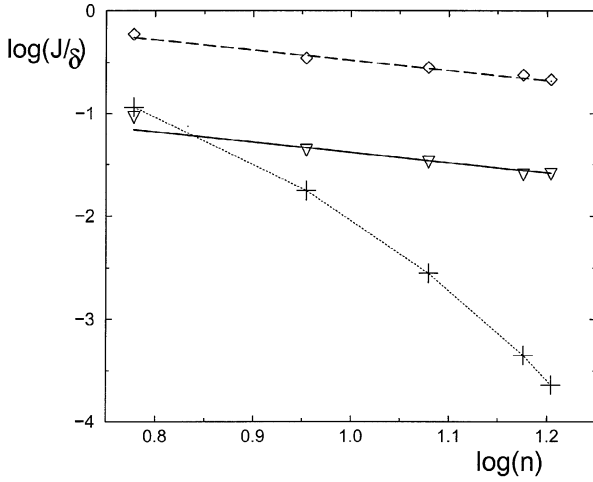


Fig. 5. Dependence of $\log(J_c/\delta)$ (diamonds) and $\log(J_{cs}/\delta)$ (triangles) versus $\log(n)$; the variation of the scaled multi-qubit spacing ($\log(\Delta_n/\delta)$) with $\log(n)$ is shown for comparison (+). Dashed line gives the theoretical formula (7) with $C_q = 3.3$; the solid line is $J_{cs} = 0.41\delta/n$; the dotted curve is drawn to guide the eye for (+). After [25].

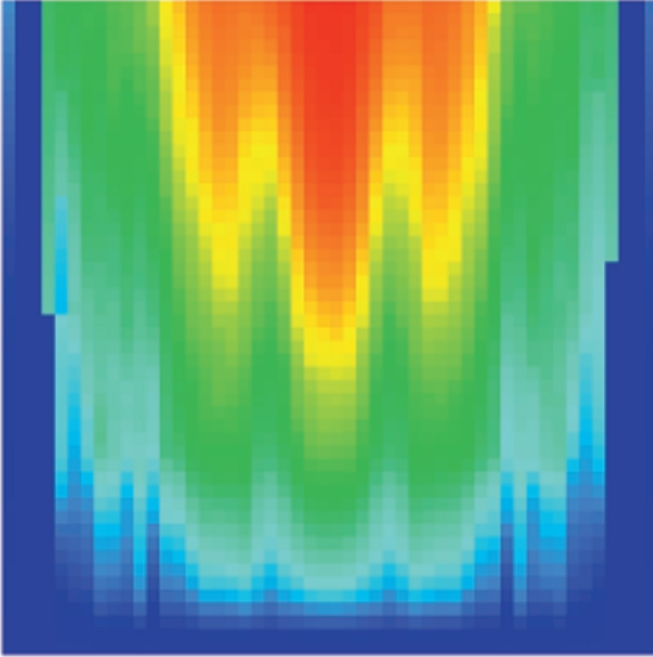


Fig. 6. The quantum computer melting induced by the coupling between qubits. Color represents the level of quantum eigenstate entropy S_q from bright red ($S_q \approx 11$) to blue ($S_q = 0$). The horizontal axis is the energy of the computer eigenstates counted from the ground state to the maximal energy ($\approx 2n\Delta_0$). The vertical axis is the value of J/A_0 , varying from 0 to 0.5. Here $n = 12$, $\delta = \Delta_0$, $J_c/\Delta_0 = 0.273$, and one random realization of (6) is chosen. After [24].

and the quantum computer hardware and its operability are destroyed by residual inter-qubit interaction.

4. Time scales for quantum chaos and decoherence in quantum computing

The results above definitely show that the optimal regime for quantum computing corresponds to the integrable region below the chaos border $J < J_c$. However, it is possible that for certain experimental proposals it will be difficult to avoid the quantum chaos regime $J > J_c$. Therefore, it is also important to understand during what time scale the quantum chaos becomes completely developed. As for TBRIM this chaotic time scale τ_χ is given by the decay rate Γ from one quantum register state to all others [24,25]:

$$\tau_\chi \approx 1/\Gamma, \quad \Gamma \sim J^2n/\delta, \quad \Gamma \sim J\sqrt{n} \quad (8)$$

where the expressions for Γ are given for $J < \delta$ and $J > \delta$ respectively. For $\delta = \Delta_0$ a similar estimate for τ_χ was given in [50].

The detailed numerical studies of chaotic disintegration of an initial state $|\chi(t=0)\rangle = |\psi_{i_0}\rangle$, corresponding to the quantum register state i_0 , are done in [25]. There for $J > J_c$ the behavior of the projection probability $F_{i_0}(t) = |\langle \psi_{i_0} | \chi(t) \rangle|^2$ is found and it is shown that the probability to stay at the initial state $F_{i_0}(t)$ decays rapidly to zero with the time scale $\tau_\chi \approx 1/\Gamma$ (see Fig. 14 there). The growth of the quantum entropy $S(t) = -\sum_i F_{i_0}(t) \log_2 F_{i_0}(t)$ with time is shown in Fig. 7. It clearly shows that after a finite time comparable with τ_χ exponentially many states are excited and the computer operability is quickly destroyed.

Above we discussed the emergence of quantum chaos induced by inter-qubit coupling in an isolated quantum computer. It is possible to assume that this process can model to a certain extent the effects of decoherence induced by coupling to the external world. Indeed, the two-body nature of interaction is also valid for external coupling. In addition

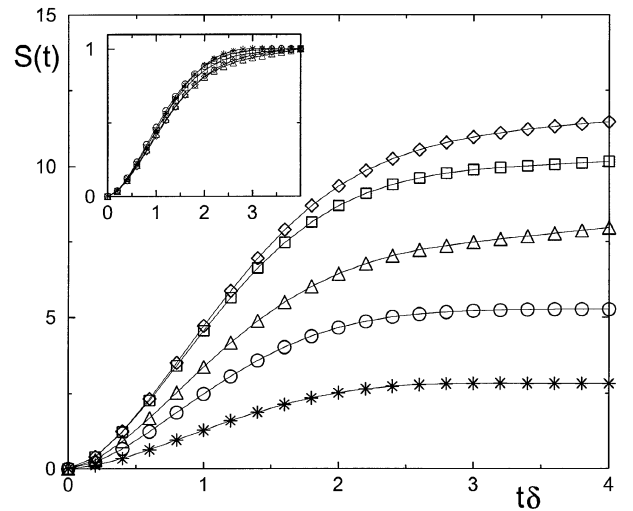


Fig. 7. Time-dependence of the quantum entropy $S(t)$ for $J/\delta = 0.4 > J_c/\delta$ and $n = 16$ (diamonds), $n = 15$ (squares), $n = 12$ (triangles), $n = 9$ (circles), $n = 6$ (*). Average is made over 200 initial states i_0 randomly chosen in the central band. Insert shows the same curves normalized to their maximal value. After [25].

it is also important to discuss another type of danger for quantum computing which is not necessary related to mixing and complex structure of eigenstates. Indeed, the algorithms constructed for quantum computing, e.g. Shor's or Grover's algorithms, are based on ideal qubits which have $\delta = 0$. However, for $\delta > 0$, and even if $J = 0$, there is a phase difference $\Delta\phi$ between different states in one band which grows with time as $\Delta\phi \sim \Delta E t$, where ΔE is the energy difference between states in the band and its maximal value can be estimated as $\Delta E \sim \delta\sqrt{n}$. It is natural to assume that as soon as the accumulated phase difference becomes comparable to 1 the computational errors become too large for correct quantum simulations. Therefore, in addition to the chaotic time scale τ_χ , there exists the phase coherence time scale τ_ϕ which is determined by the condition $\Delta\phi(\tau_\phi) \sim \Delta E \tau_\phi \sim 1$. This scale is finite even at $J = 0$ and can be estimated as $\tau_\phi \sim 1/(\delta\sqrt{n})$ since the band width is $\Delta E \sim \delta\sqrt{n}$ and the computer usually operates with all states inside the band. For small J this estimate is still valid and in this case $\tau_\phi < \tau_\chi$ for $J < \delta n^{1/4}$. On the contrary for $\delta = 0$ and $J > 0$ the energy band width is $\Delta E \sim J\sqrt{n}$ and according to (8) both scales are comparable $\tau_\phi \sim \tau_\chi$. While both time scales τ_χ and τ_ϕ are important for the quantum computer operability it is clear that the effects of quantum chaos on the scale τ_χ are much more dangerous since after this scale an exponentially large number of quantum register states are mixed as it is shown in Fig. 7. On the contrary on the scale τ_ϕ only the phases are changed but not the number of states. Therefore, it is natural to expect that phase spreading can be more easily suppressed by error-correcting codes than the onset of quantum chaos.

These results show that there are two important time scales τ_χ and τ_ϕ . The effect of external noise and decoherence can be also characterized by two different scales. Indeed, from one side the noise gives some effective rate Γ_T with which a multi-qubit state decays to other states that determines the time scale $\tau_\chi \sim 1/\Gamma_T$. It is clear that $\Gamma_T \propto n$ since the noise acts on all qubits independently. At the same time the noise gives some effective diffusion in energy which can be estimated as $D_E = \omega^2 \Gamma_T$ where ω is a typical energy change induced by noise during the time $1/\Gamma_T$. For example, noisy fluctuations of J_{ij} in (6) give transitions with an energy change $\omega \sim \delta$. As a result, $\Delta E \sim \sqrt{D_E \tau_\phi}$ and $\Delta\phi \sim D_E^{1/2} \tau_\phi^{3/2} \sim 1$. This determines two time scales for external decoherence:

$$\tau_\chi \approx 1/\Gamma_T, \quad \tau_\phi \approx 1/(\omega^2 \Gamma_T)^{1/3} \quad (9)$$

where τ_χ is related to the decay and relaxation, while τ_ϕ determines the phase coherence. Let us note that the phase decoherence was discussed for electrons in metals at low temperature [51] and was observed experimentally, see e.g. [52]. However, there Γ_T was independent of the number of electrons, while for quantum computing Γ_T is proportional to the number of qubits since the global coherence of multi-qubit states should be preserved.

The time scales (9) play an important role for quantum computing. Indeed, the gate operations should be fast enough compared to these scales to allow to realize error-correcting codes [7,8] and to avoid a destruction of operability after the time scales (9). Here, it is important to stress an important property of quantum evolution for

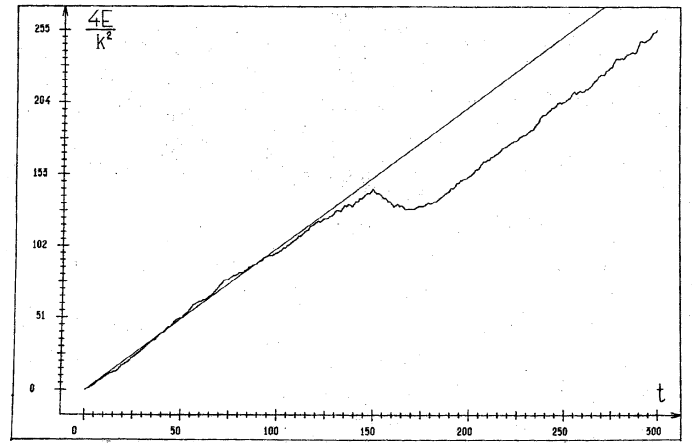


Fig. 8. Time-dependence of the energy $E = \langle n^2/2 \rangle$ in the Chirikov standard map (10) in the chaotic regime with $K = 5$, $k = 20$ for 1000 orbits homogeneously distributed at $n = 0$ for $t = 0$. The straight line shows the theoretical diffusion $E = k^2 t / 4$. All velocities are inverted after $t = 150$ with computer accuracy 10^{-12} that destroys time reversibility after 20 iterations. After [53].

which there is no exponential growth of errors, contrary to the classical dynamics. An illustration of this fact is based on the dynamics of the Chirikov standard map [53]:

$$\bar{n} = n + k \sin(\theta + Tn/2), \quad \bar{\theta} = \theta + T(n + \bar{n})/2. \quad (10)$$

Here, (n, θ) is a pair of momentum and phase variables and bar denotes the new values of variables after one period of perturbation. The classical dynamics depends only on the chaos parameter $K = kT$ and for $K > 0.9716..$ the dynamics becomes globally diffusive in n . In this regime the classical trajectories are exponentially unstable and have positive Kolomogorov entropy. Due to the fact that the computer errors grow exponentially quickly in time that in practice destroys the time-reversibility of the map (10). This fact is illustrated in Fig. 8 [54] where the iterations are done on a computer with errors of the order of 10^{-12} . Due to exponential instability of classical orbits the time-reversibility for orbits with inverted momentum ($n \rightarrow -n$) completely disappears after 20 iterations. The situation is completely different for the corresponding quantum dynamics described by the unitary evolution operator for wave function ψ on one iteration [54]:

$$\bar{\psi} = \exp(-iT\hat{n}^2/4) \exp(-ik \cos \theta) \exp(-iT\hat{n}^2/4) \quad (11)$$

where $\hat{n} = -i d/d\theta$ and $\hbar = 1$. Here the quantum dynamics simulated on the same computer remains exactly reversible as it is shown in Fig. 9.

The above results clearly show that there is no exponential instability in quantum mechanics [54]. This gives a direct indication that the errors in quantum computing can be efficiently corrected since e.g. Shor's algorithm requires relatively small numbers of gate operations N_G for number factorization: $N_G \sim 300L^3$ where L is the number of digits in the number to be factorized [5]. In a sense the Shor algorithm is exponentially fast while the errors grow only as a power of time. This opens broad perspectives for quantum computing but requires a development of efficient error-correcting codes. There is also another general question related to the uncertainty relation between energy and time.

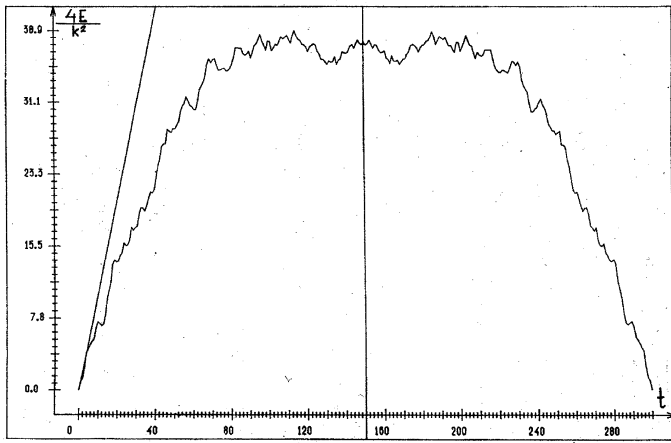


Fig. 9. Same as in Fig. 8 but for the corresponding quantum system (11) with $K = 5$, $k = 20$, $\hbar = 1$ realized on the same computer. The vertical line marks the inversion of time made by replacement $\psi \rightarrow \psi^*$. The quantum dynamics remains perfectly reversible. After [53].

Indeed, the quantum computing during the time Δt allows to resolve energy levels only on the energy scale $\Delta E \sim 1/\Delta t$. This means that the resolution of exponentially small spacing $\Delta E \sim \Delta_n$ requires exponentially long times analogous to the Heisenberg time scale in the quantum chaos $t_H \sim 1/\Delta_n$. Apparently only after this time scale all exponentially large information hidden in the Hilbert space can become available. However, it is possible that on shorter time scales useful information can be extracted in a way unaccessible to classical computers, e.g. Shor's factorization. But the question about how many such exponentially efficient algorithms can be found remains open.

5. Conclusion

In this paper the conditions for emergence of quantum chaos, ergodicity and dynamical thermalization in many-body quantum systems with interaction and disorder are presented. The Åberg criterion represents the main condition for onset of quantum chaos and different checks performed for various physical systems confirm its validity. The generalization of these results allows to determine the quantum chaos border for quantum computing. In particular, it is shown that the critical coupling between qubits, which leads to quantum chaos and quantum computer hardware melting, drops only linearly with the number of qubits and is exponentially larger than the energy level spacing between eigenstates of the quantum computer. In this sense the ideal multi-qubit structure is rather robust with respect to perturbations which opens broad possibilities for realization of quantum computers.

Of course, the optimal regime for quantum computer operability corresponds to the integrable regime below the quantum chaos border. In this respect the quantum chaos is a negative effect for quantum computing and should be eliminated. Here it is possible to make an analogy with the development of classical chaos. This phenomenon also has negative effects for operability of plasma traps and accelerators. In fact the very first Chirikov resonance-overlap criterion had been invented in the pioneering work [55] for the explanation of experiments on plasma confine-

ment in open magnetic traps and later found broad applications in accelerator physics [56]. Since 1959 it is the only simple physical criterion which allows to determine the chaos border in classical nonlinear hamiltonian systems [55,56,53]. Indeed, the interest to the quantum chaos appeared only after a deep understanding of classical chaos had been reached. In a similar way it is possible to think that the deep understanding of quantum chaos in many-body systems in future will allow to quantum computers operate in a better way.

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