Many-body localization edge in the random-field Heisenberg chain

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Outline

- Broad question: What happens to Anderson localization in presence of interactions?

«Many-body localization» (MBL)

A new distinct dynamical phase of matter, which does not thermalize

Part 1 : Mini-Review on Many-Body localization

- Distinct features from a «thermal» state, and Anderson insulator
- Present in simple toy models
- Many open questions…

Part 2 : «Large»-scale numerics on a MBL Hamiltonian

- Computational issues
- Energy-resolved phase diagram

my current understanding of

Part 1 : Mini-Review on Many-Body Localization

Reviews on MBL :

Nandkishore & Huse, arxiv:1404.0686

Altman & Vosk, arxiv:1408.2834

Thermalisation & ETH

- Initial wave-function $|\Psi_0\rangle = \sum_i a_n|n\rangle$ expressed in the eigenbasis of $H = \sum_n E_n|n\rangle\langle n|$
- Time-evolved observable (generic Hamiltonian)

$$\langle \mathcal{O}(t)\rangle = \sum_{n,n'} a^*_{n'} a_n e^{-i(E_{n'}-E_n)t}\mathcal{O}_{nn'} \xrightarrow{t\to\infty} \sum_n |a_n|^2 \mathcal{O}_{nn}$$

«Diagonal ensemble»

- Eigenstate thermalization hypothesis (ETH)

Deutsch, Srednicki, Rigol & many authors

$$\langle n|\mathcal{O}|n\rangle \simeq \langle n'|\mathcal{O}|n'\rangle = \mathcal{O}(E)$$ $|n\rangle, |n'\rangle$ in the same energy shell

$\langle n|\mathcal{O}|n'\rangle$ vanish in the thermodynamic limit and for local observable

- ETH + (some other minimal assumptions) implies thermalisation

$$\langle \mathcal{O}(t\to\infty)\rangle = \mathcal{O}(E) = \mathcal{O}(T)$$

$$E = \langle \Psi_0|H|\Psi_0\rangle$$

$$E = \langle H\rangle_T$$

Consequences

- Each eigenstate is thermal, «knows» equilibrium

$$\rho(0) = |n\rangle\langle n| = \rho(t) = \rho^{eq}(T_n) \qquad E_n = \langle H \rangle_{T_n}$$

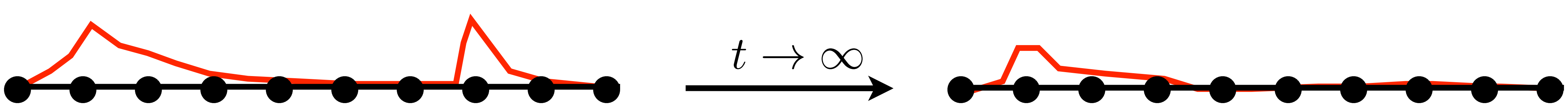
- Memory of initial conditions is lost
- ETH is a «justification» of the microcanonical ensemble at the invididual eigenstate level

- ETH seems to work (analytics+numerics) for most many-body quantum systems, except

- Integrable systems : May have their own ETH, relaxation to a Generalized Gibbs Ensemble

many authors...

- Localized systems : single-particle localization



- Many-Body Localized systems

Many-body localization

- Old problem revived by an enormous amount of contributions!

  Anderson, Fleishmann, Shepelyansky...

  Nandkishore, Huse, arxiv:1404.0686, Altman, Vosk 1408.2834

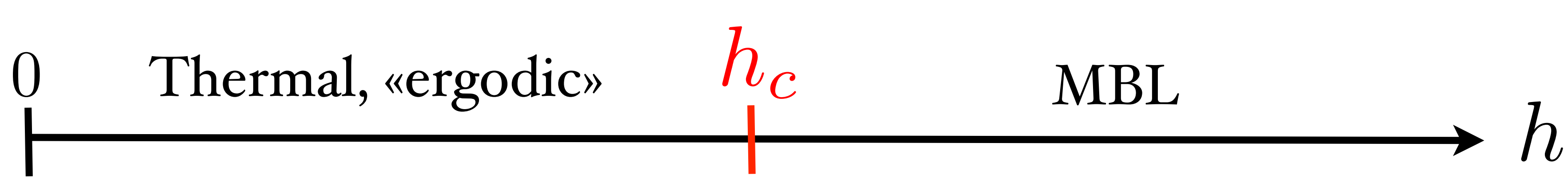
- Typical example : XXZ chain with random fields

$$H = \sum_i \sigma_i^z \sigma_{i+1}^z + \sigma_i^z \sigma_{i+1}^z + \Delta \sigma_i^z \sigma_{i+1}^z - \sum_i h_i \sigma_i^z \qquad h_i \in [-h, h]$$

- Infinite disorder : eigenstates are fully localized product-states, no entanglement
- Branch small interaction : perturbative calculations (Gornyi et al. Basko et al.) indicate that

  thermalization does not occur: states keep localized, no spin or energy transport

- Beyond perturbation : numerics (including this talk) indicate that the localized phase survives



Phase transition (?) is dynamical
can't be seen in thermodynamics

- Crucial to work in the «eigenstate ensemble», not (micro-)canonical ensemble

Phenomenology of MBL systems

- Consider only fully MBL systems (all eigenstates are localized)
- Exact results Imbrie, phenomenology Huse. Oganesyan, Abanin *et al.*, perturbative results, Ros, Müller Scardicchio strong disorder RG Altman, Vosk

  - Idea: Quasi-local unitary transform can «diagonalize» the Hamiltonian

$$U^\dagger H U = -\sum_i h_i \tau_i^z - \sum_{i<j} J_{i,j}\tau_i^z\tau_j^z + \sum_{i<j<k} J_{i,j,k}\tau_i^z\tau_j^z\tau_k^z + \ldots,$$

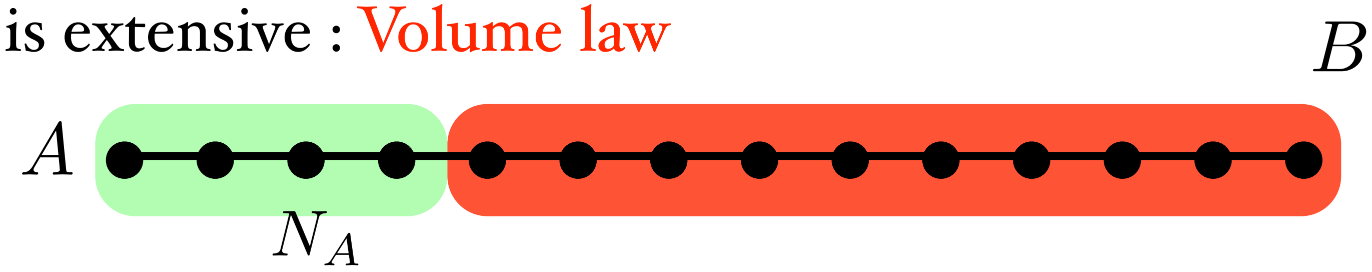
$J_{i_1,\ldots,i_k}$ decay exponentially

- Set of localized bits: $\tau_i^z \simeq \hat{Z}_i(h)\sigma_i^z + \mathrm{tail}$
- $U\tau_i^z U^\dagger$ = complete set of local integral of motions

- Useful to describe properties of MBL, to detect MBL through spin-echo experiments Serbin *et al.*
- Other sets of local integrals of motion can be constructed, which may have a better physical interpretation Ros, Müller Scardicchio Abanin *et al.*

Entanglement & MBL

- ETH : Entanglement entropy of eigenstates is extensive : Volume law



$$\rho_A = \mathrm{Tr}_B \rho = \rho_A^{\mathrm{eq}}(T_n)$$

$$S_A = -\mathrm{Tr}_A \rho_A \log \rho_A \propto N_A \ \text{ if } \ T_n \neq 0$$

System is its own bath: B acts a thermal bath for A

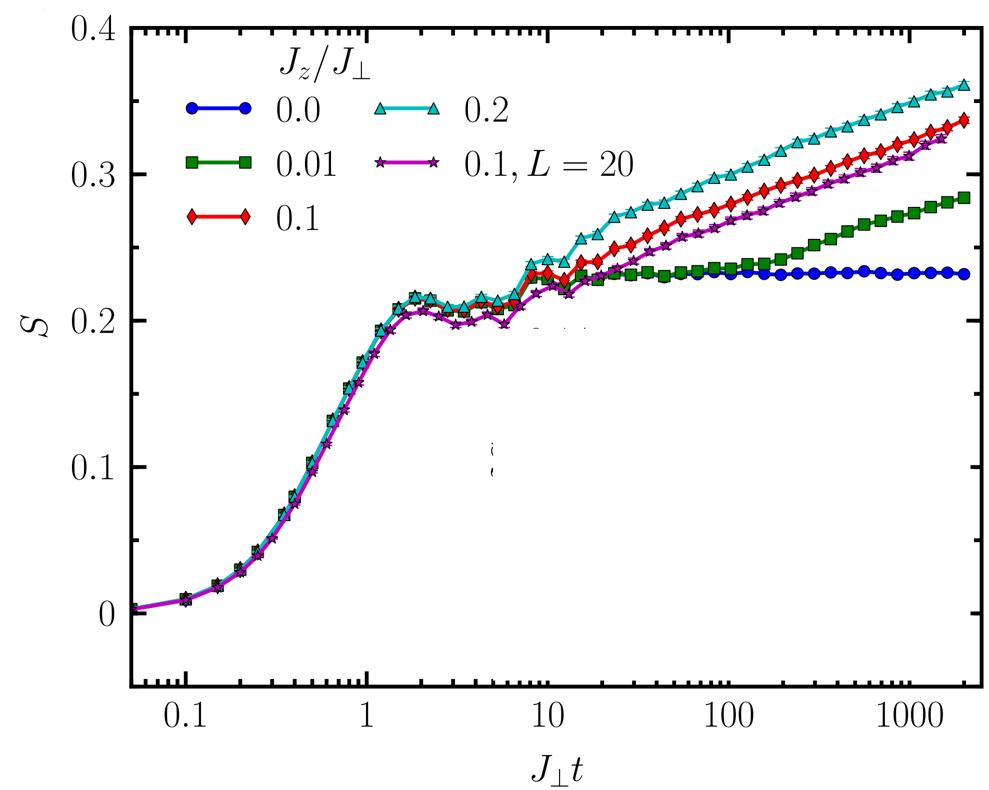
- MBL states have low entanglement $S_A / N_A \to 0$
  - MBL states efficiently represented as matrix-product states
  - Entanglement spreads logarithmically

Eisert *et al.*

Abanin, Vidal *et al.*

Pekker, Clark

Can be understood with the localized-bits picture



Znidaric *et al.*

Bardarson *et al.*

Summary of MBL

from Nandkishore, Huse arxiv:1404.0686,

Thermal phase	Single-particle localized	Many-body localized
Memory of initial conditions 'hidden' in global operators at long times	Some memory of local initial conditions preserved in local observables at long times	Some memory of local initial conditions preserved in local observables at long times.
ETH true	ETH false	ETH false
May have non-zero DC conductivity	Zero DC conductivity	Zero DC conductivity
Continuous local spectrum	Discrete local spectrum	Discrete local spectrum
Eigenstates with volume-law entanglement	Eigenstates with area-law entanglement	Eigenstates with area-law entanglement
Power-law spreading of entanglement from non-entangled initial condition	No spreading of entanglement	Logarithmic spreading of entanglement from non-entangled initial condition
Dephasing and dissipation	No dephasing, no dissipation	Dephasing but no dissipation

- MBL also found in quasi-periodic systems
- MBL states can host «forbidden» (discrete-symmetry breaking, topological) order, in 1d at finite E
- Coupling with a bath: MBL physics can still be detected

Some open questions

- MBL in systems with no disorder? Müller *et al.*, Grover & Fisher; De Roeck & Huveneers, Garrahan *et al.*, Yao *et al.*
- Many-body mobility edge: Griffiths effects? sub-diffusive ergodic phase? Agarwal *et al.*
- Nature of many-body localization transition?

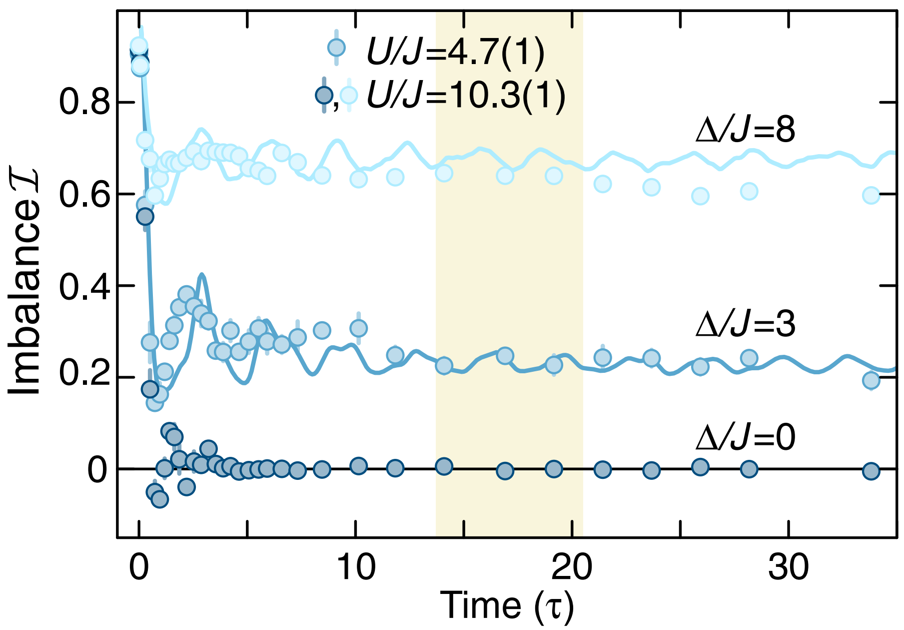
Experimental realization

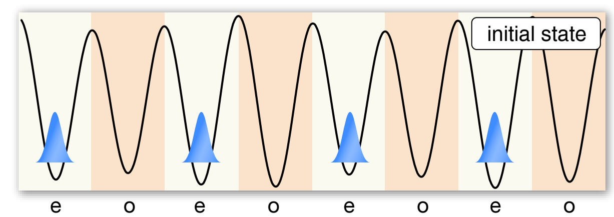
Schreiber *et al.*, arXiv:1501.05661

- Cold-atomic gas realization of interacting Aubry-André model:

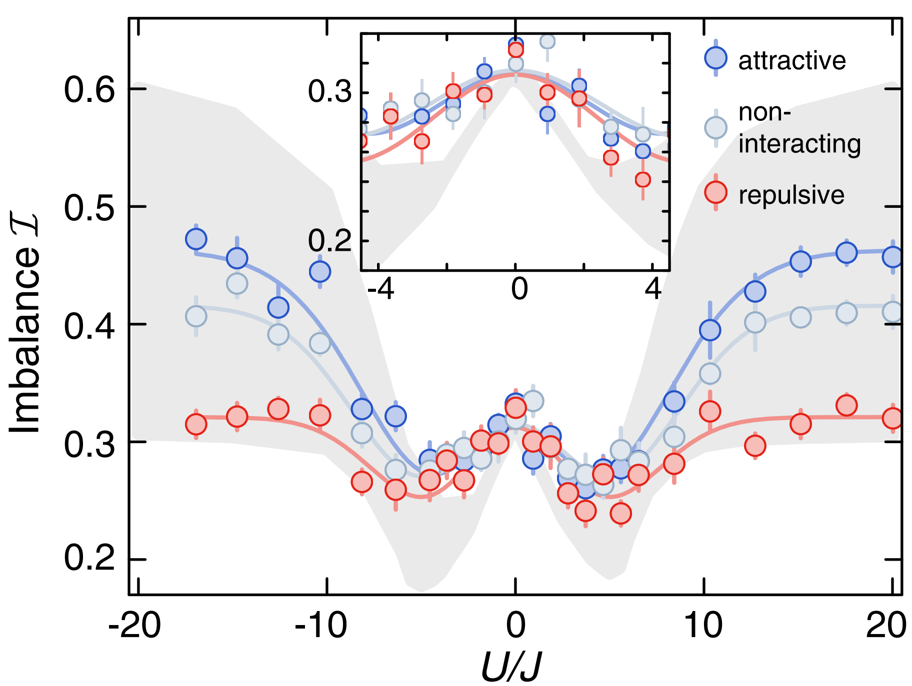
$$\hat{H} = -J\sum_{i,\sigma}\left(\hat{c}^{\dagger}_{i,\sigma}\hat{c}_{i+1,\sigma} + \text{h.c.}\right) + \Delta\sum_{i,\sigma}\cos(2\pi\beta i + \phi)\hat{c}^{\dagger}_{i,\sigma}\hat{c}_{i,\sigma} + U\sum_{i}\hat{n}_{i,\uparrow}\hat{n}_{i,\downarrow}.$$

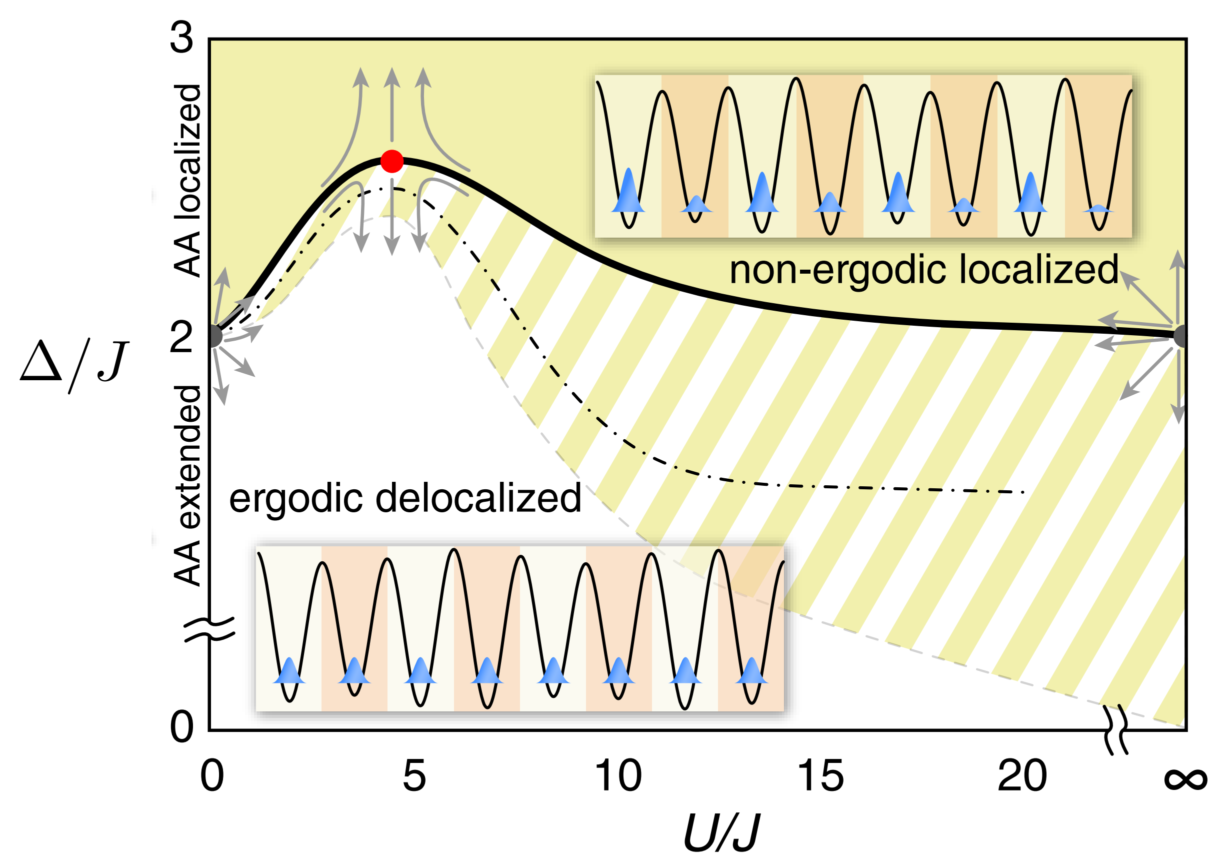
- (Non-)Equilibration of a quenched initial state measured by imbalance





$$\mathcal{I} = \frac{N_e - N_o}{N_e + N_o}$$





Part 2 : «Large»-scale numerics on a MBL Hamiltonian

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- How to detect MBL states in numerics?
- Presence of a many-body mobility edge?
- Nature of the MBL transition? First fingerprints of universality class…
- Is MBL a true localization in Hilbert space?

MBL & Numerics

- Prototypical MBL Hamiltonian : Heisenberg S=1/2 spin chain in a random field

$$H = \sum_{i=1}^{L} \mathbf{S_i} \cdot \mathbf{S}_{i+1} - \sum_i h_i S_i^z \qquad h_i \in [-h, h]$$

- Total magnetization S^z is conserved, no other symmetries
- A tough computational problem
  - Almost no symmetries, average over disorder
  - MBL physics is located at high-energy: eigenstates in the middle of the spectrum
  - Ground-state methods are not well adapted

    ~~DMRG, power method, Lanczos, T=0 series expansion, etc~~

  - We want eigenstates of a closed system: cannot impose a bath !

    ~~Quantum Monte Carlo, high-T series expansion ...~~

- We are left with
  - Time evolution after a quench: time-dependent DMRG, TEBD ... (but no eigenstates)
  - Real Space Renormalization Group: for excited states, only in the strong disorder limit
  - Brute-force methods: Full diagonalization, or (slightly better) spectral transforms

Computational details

- Obtain eigenstates in the middle of the spectrum with shift-invert

$$H \longrightarrow R = (H - \sigma I) \longrightarrow R^{-1}$$

- Extremal eigenvalue problem for R^{-1} (Lanczos-like methods …)
- We cannot compute R^{-1} ! However just need to know how to apply $R^{-1}a = b$
- Three steps method:
  - LU decomposition $R = LU$ Bottleneck !
  - Solve for c in $a = Lc$
  - Solve for b in $c = Ub$

  «Simple» linear equations

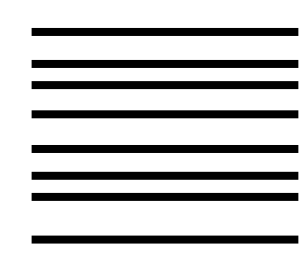
- LU decomposition done by the massively parallel solver Mumps

Computational details

- Obtain eigenstates in the middle of the spectrum with shift-invert

$$H \longrightarrow R = (H - \sigma I) \longrightarrow R^{-1}$$

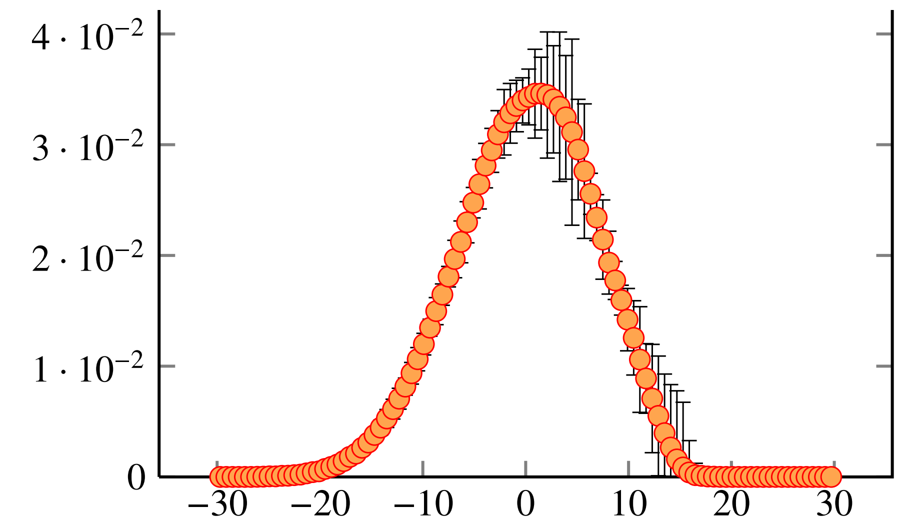
- Further details:
  - Method typical for Anderson localization, however LU much more difficult here !
  - L=22 (matrix size ~ 700.000) instead of L=16 (matrix size ~ 12.000)
  - Obtain ~ 50 eigenstates for ~ 1000 disorder realizations for each field, energy, L
  - Strong correlations between eigenstates of the same disorder sample!
  - All results in terms of $\epsilon = (E - E_{\text{max}})/(E_{\text{min}} - E_{\text{max}})$



Gap ratios

- Level statistics: natural tool to check for localization
  - Thermal (ETH) phase: expect Random Matrix Theory (in particular GOE) to correctly capture highly-excited eigenvalues
  - MBL phase: expect Poisson statistics (no correlation, no level repulsion)

- However, unfolding necessary due to d.o.s. effects



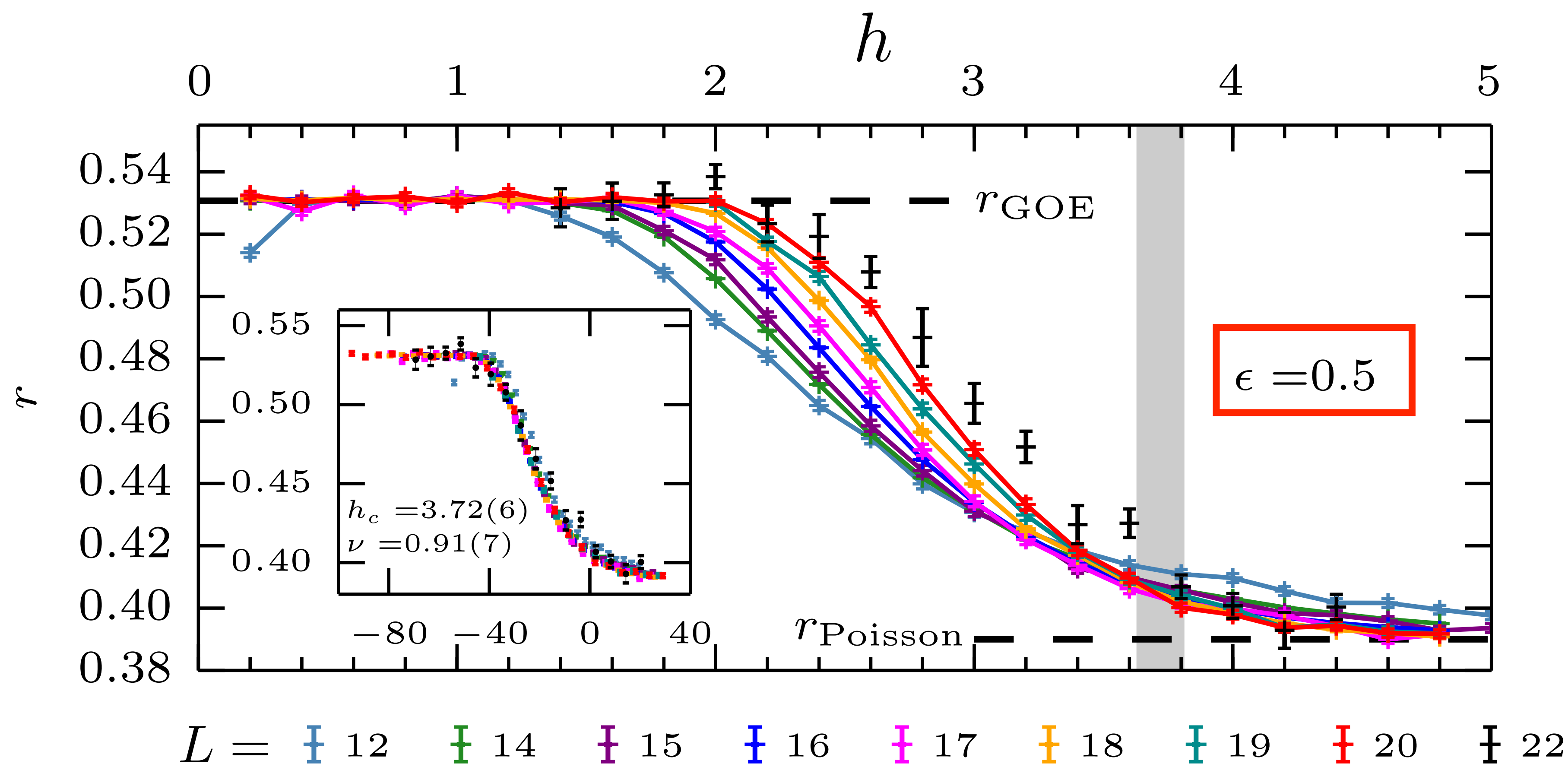
- Gap ratio Oganesyan, Huse

$$g_n = |E_n - E_{n-1}| \qquad r = \min(g_n, g_{n+1}) / \max(g_n, g_{n+1})$$

$$\langle r \rangle_{\text{GOE}} \simeq 0.5307 \qquad \langle r \rangle_{\text{Poisson}} \simeq 0.3863$$

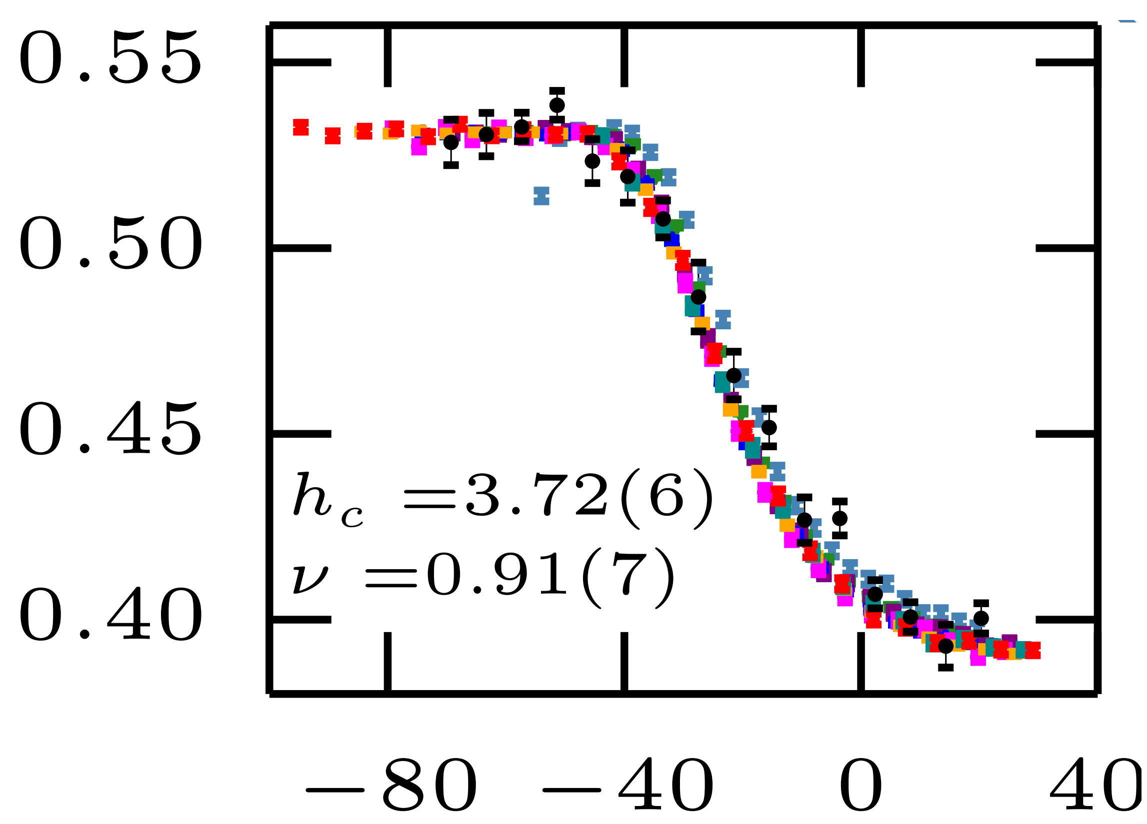
Gap ratios

- Energy-resolved data



- Finite-size scaling ansatz

$$r = f_r(|h - h_c|.L^{1/\nu})$$



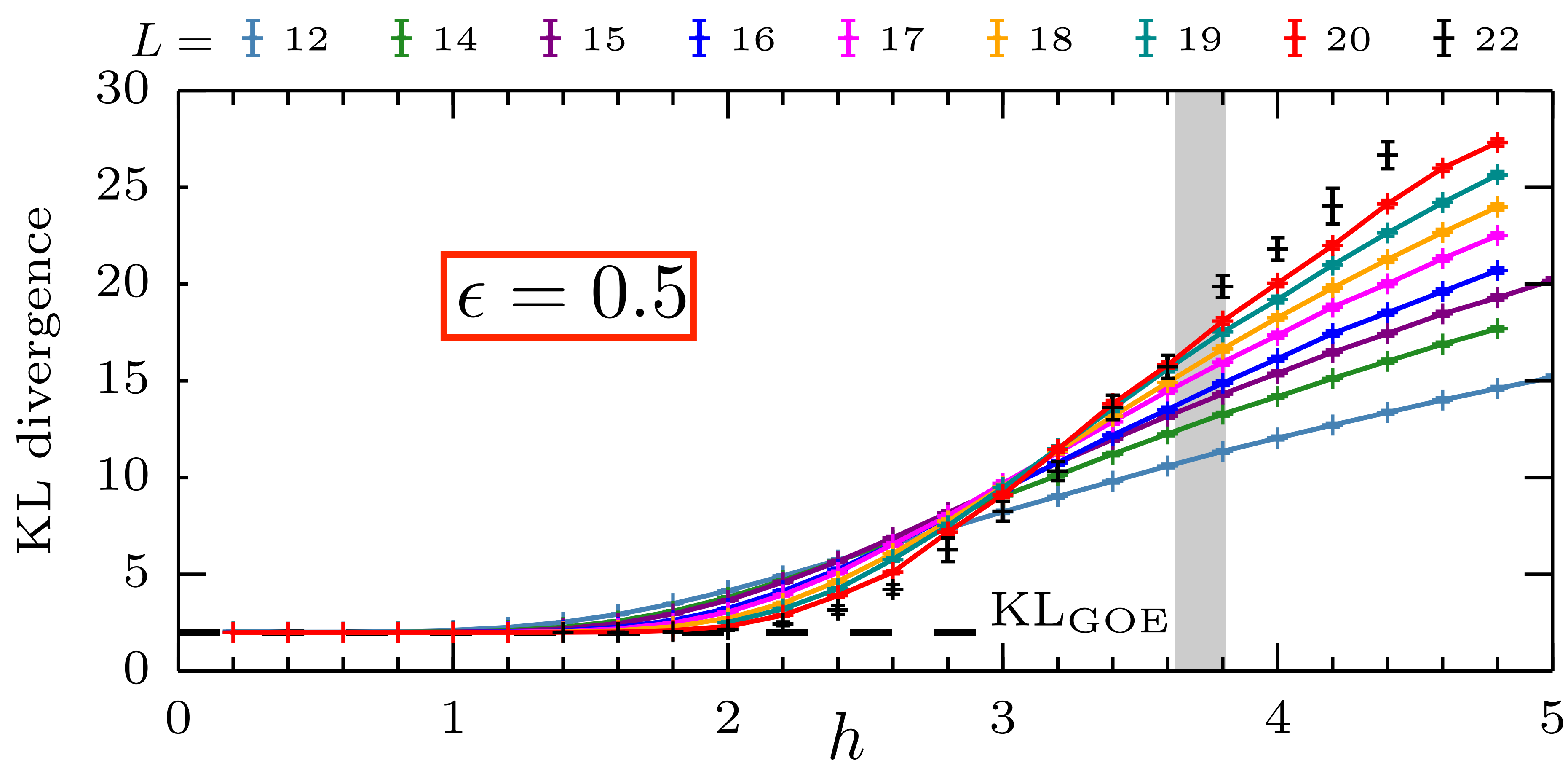
More...

Eigenstate correlations

- Beyond level statistics: correlations between eigenstates
  - Thermal (ETH) phase: expect eigenstates to be «similar»
  - MBL phase: expect eigenstates to be «different»
- Kullback-Leibler divergence quantify similarity between eigenstates (in a basis)

$$\mathrm{KL} = \sum_i p_i \ln(p_i/q_i) \qquad \begin{aligned} p_i &= |\langle n|i\rangle|^2 \\ q_i &= |\langle n'|i\rangle|^2 \end{aligned} \qquad \{|i\rangle\} = \{S^z\} \text{ basis}$$

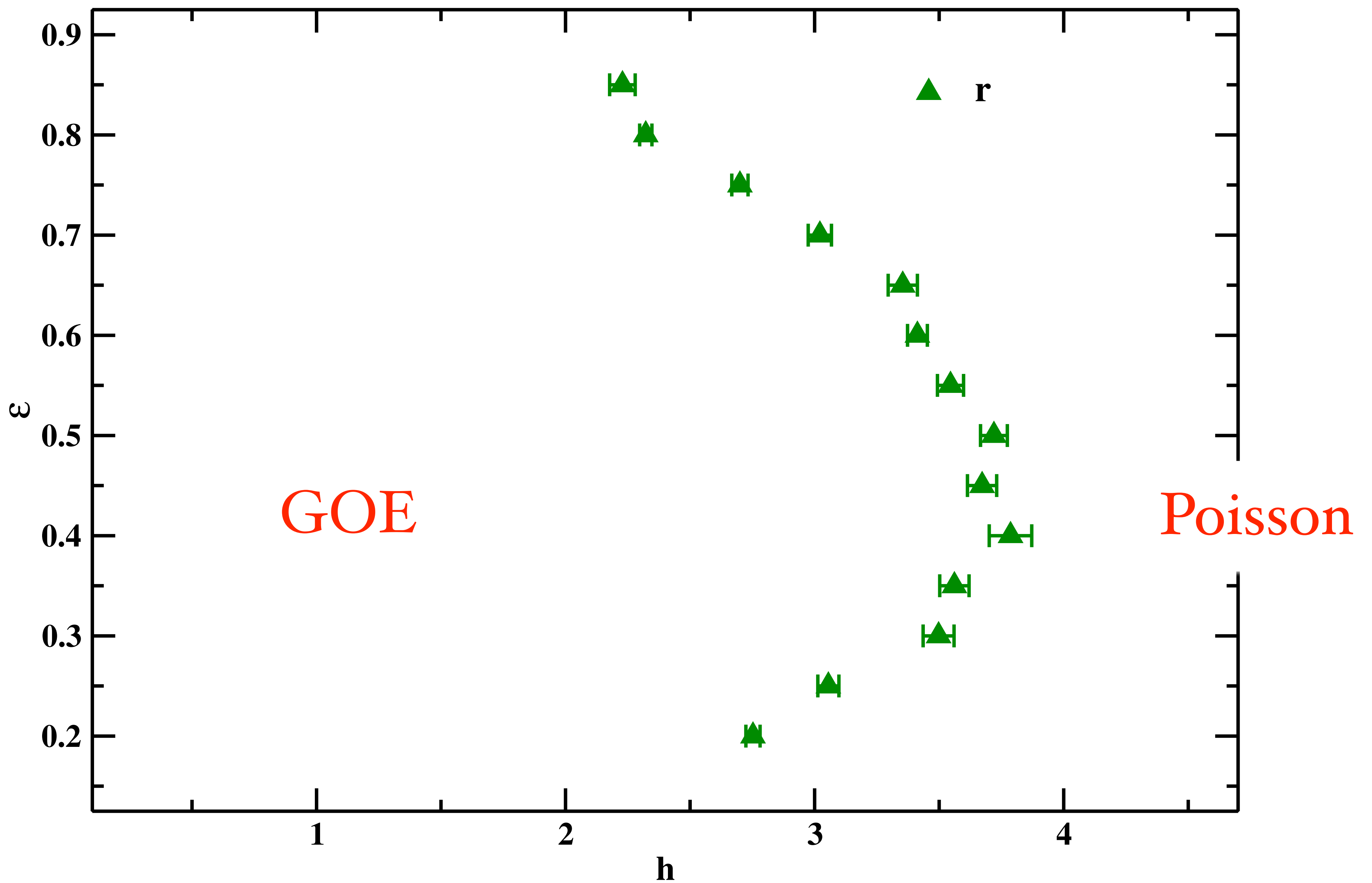
- Identical states : $\mathrm{KL} = 0$, GOE : $\mathrm{KL} = 2$, diverges for very different states



More...

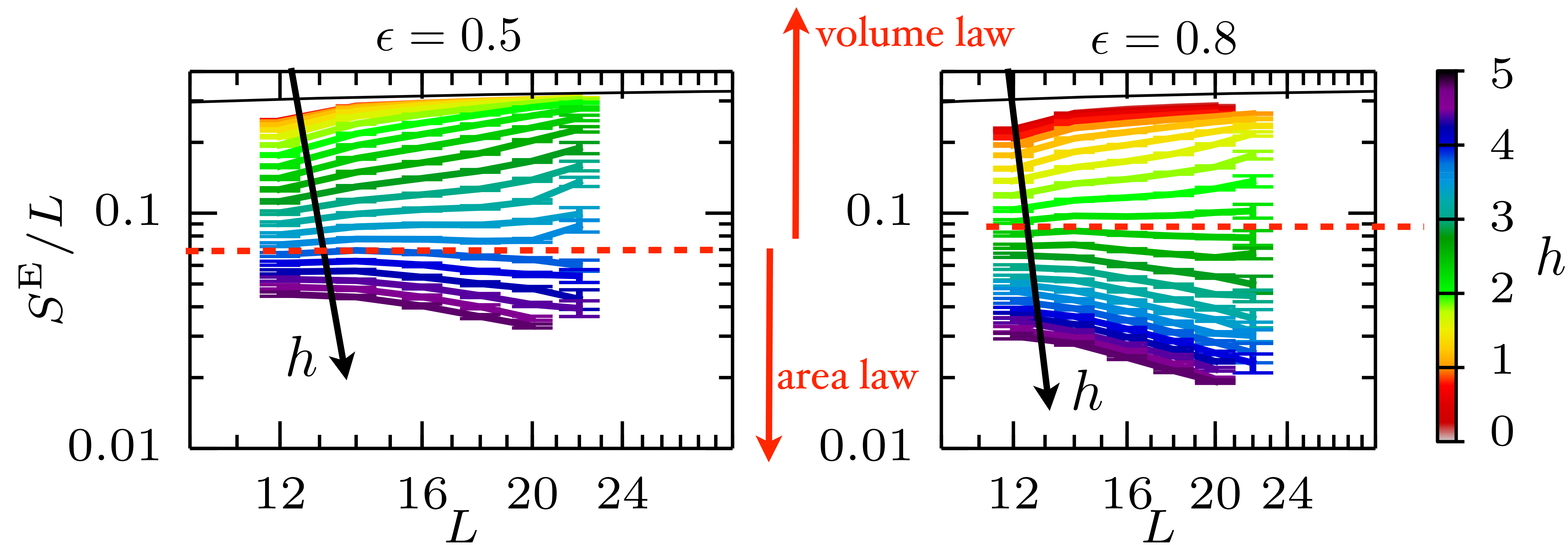
Phase diagram (1)

- Energy-resolved phase diagram

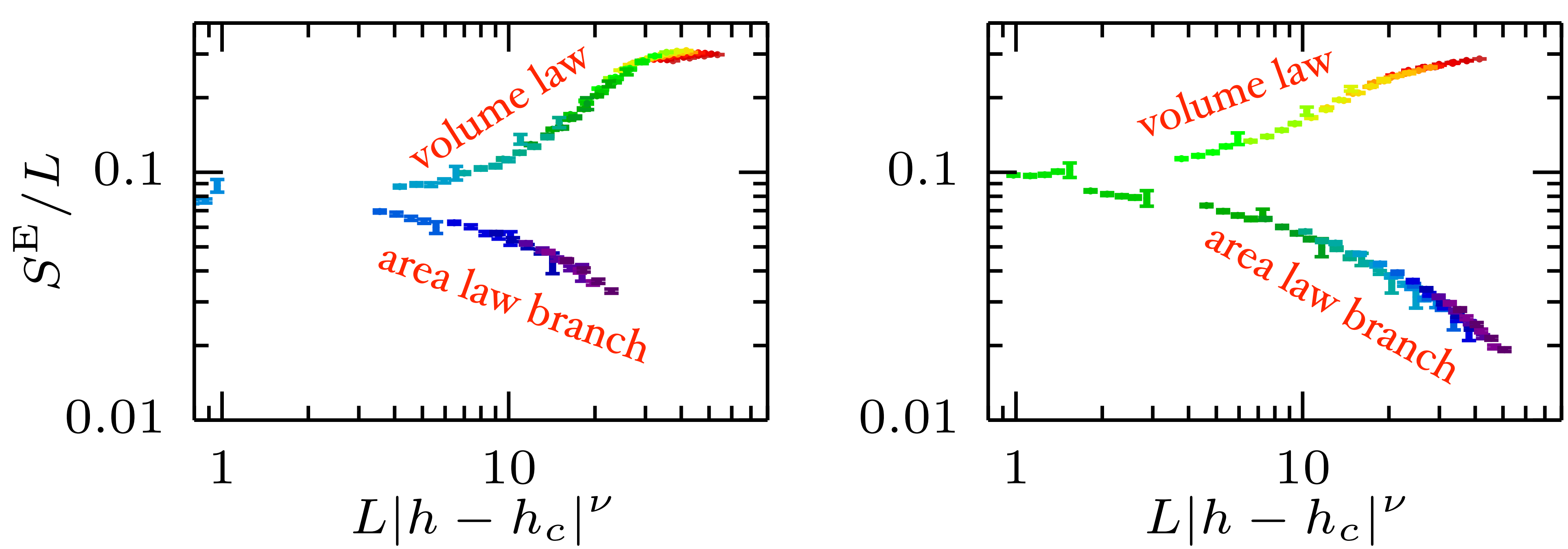


Entanglement entropy

- Area vs. volume law scaling of entanglement entropy distinguishes the two phases

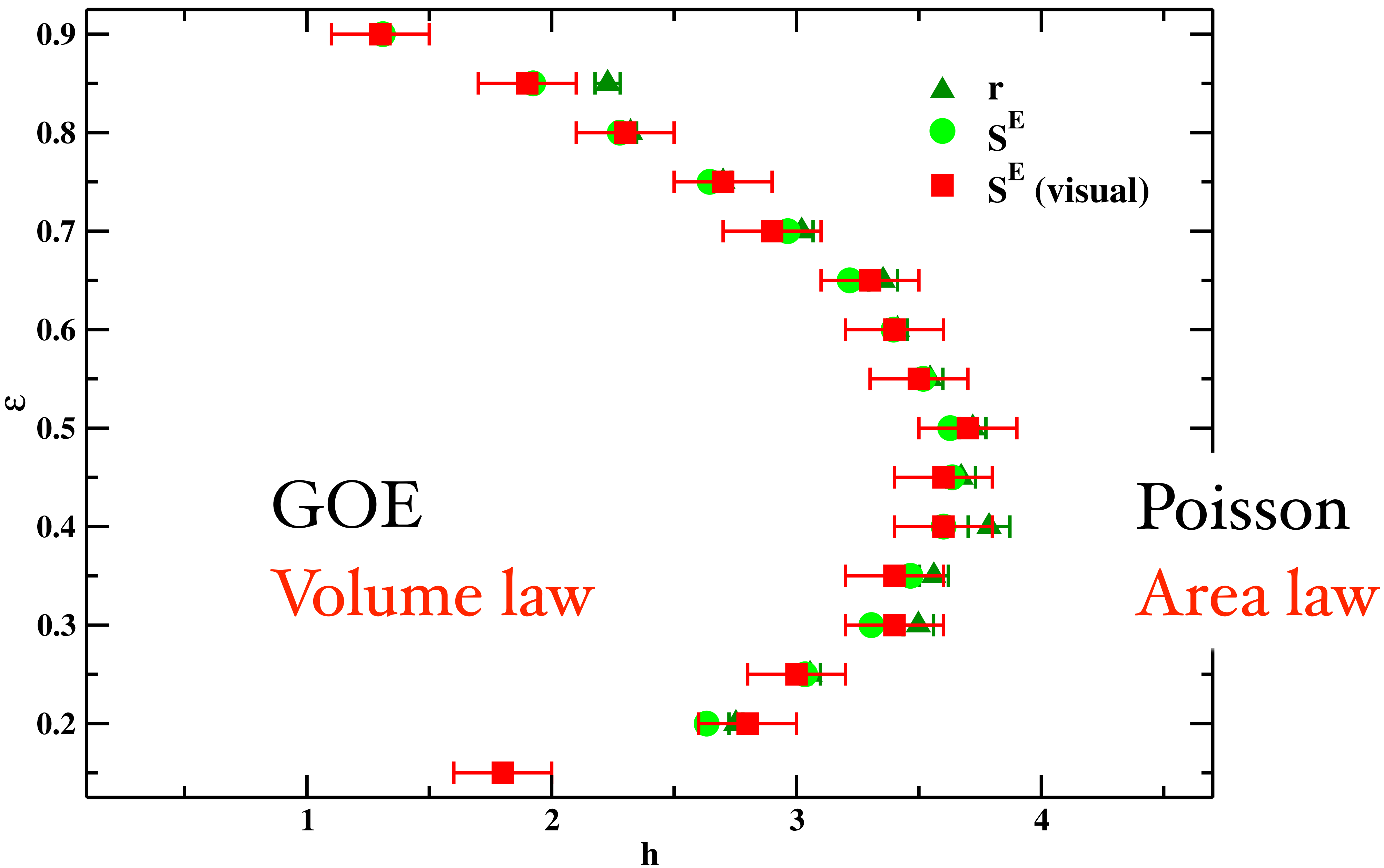


- Collapse to finite-size scaling form: $S^{\mathrm{E}}/L = f_S(|h-h_c|.L^{1/\nu})$



Phase diagram (2)

- Energy-resolved phase diagram



Memory of initial magnetization

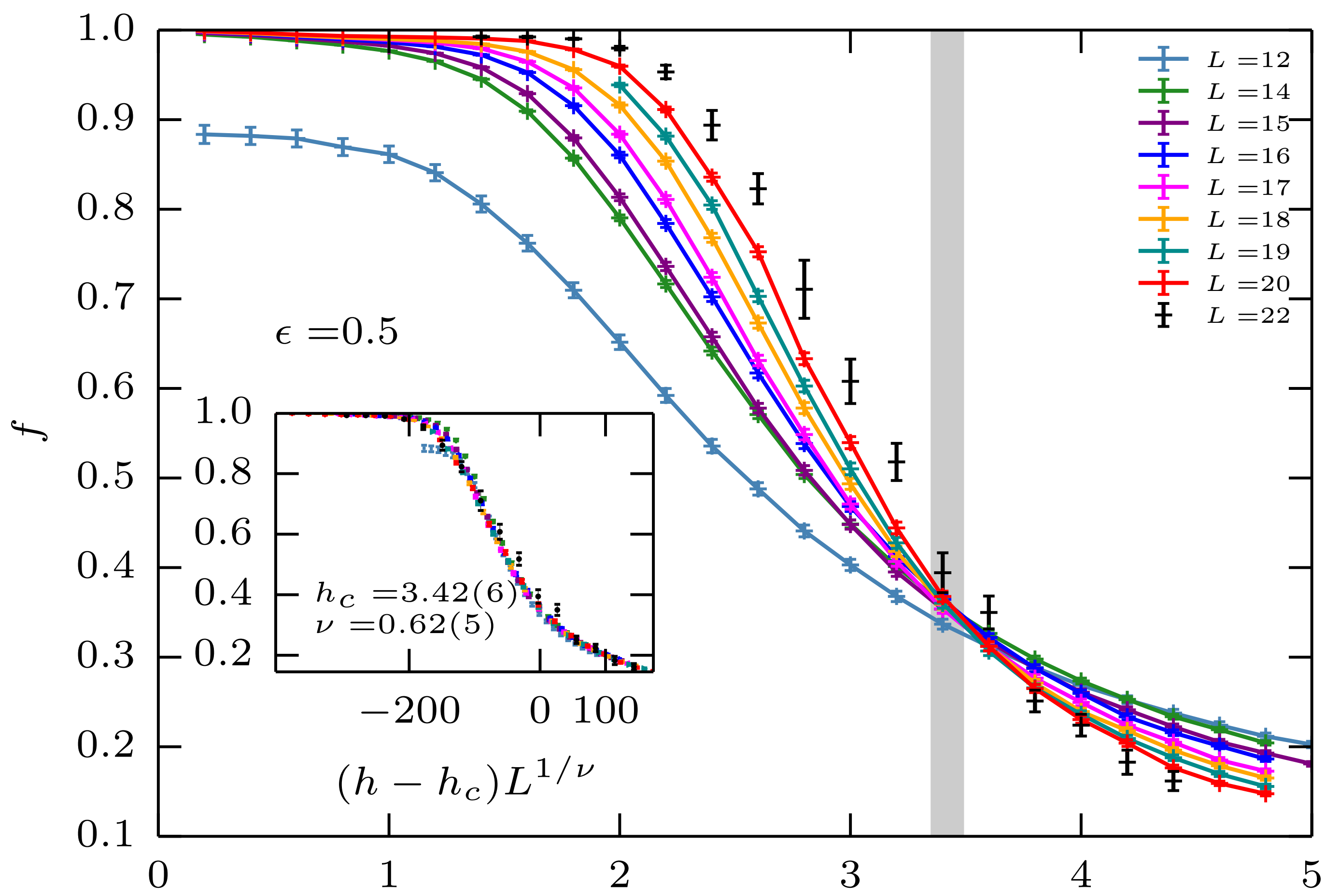
following Pal & Huse

- Consider relaxation of an initial spin inhomogeneity $\hat{M} = \sum_r S_r^z e^{i2\pi r/L}$
- Prepare the initial state $\rho_0 = (1 + x\hat{M}^\dagger)/Z$
- Initial spin polarization $\langle \hat{M} \rangle_0 = \mathrm{Tr}\rho_0 \hat{M} = \frac{x}{Z} \sum_n \langle n|\hat{M}^\dagger \hat{M}|n\rangle$
- Final spin polarization $\langle \hat{M} \rangle_\infty = \sum_n \rho_0^{nn} \langle n|\hat{M}|n\rangle = \frac{x}{Z} \sum_n \langle n|\hat{M}^\dagger|n\rangle \langle n|\hat{M}|n\rangle$

diagonal ensemble

- Contribution of eigenstate $|n\rangle$ to depolarization $f_n = 1 - \frac{\langle n|\hat{M}^\dagger|n\rangle\langle n|\hat{M}|n\rangle}{\langle n|\hat{M}^\dagger \hat{M}|n\rangle}$

$f = 1$
«No memory»

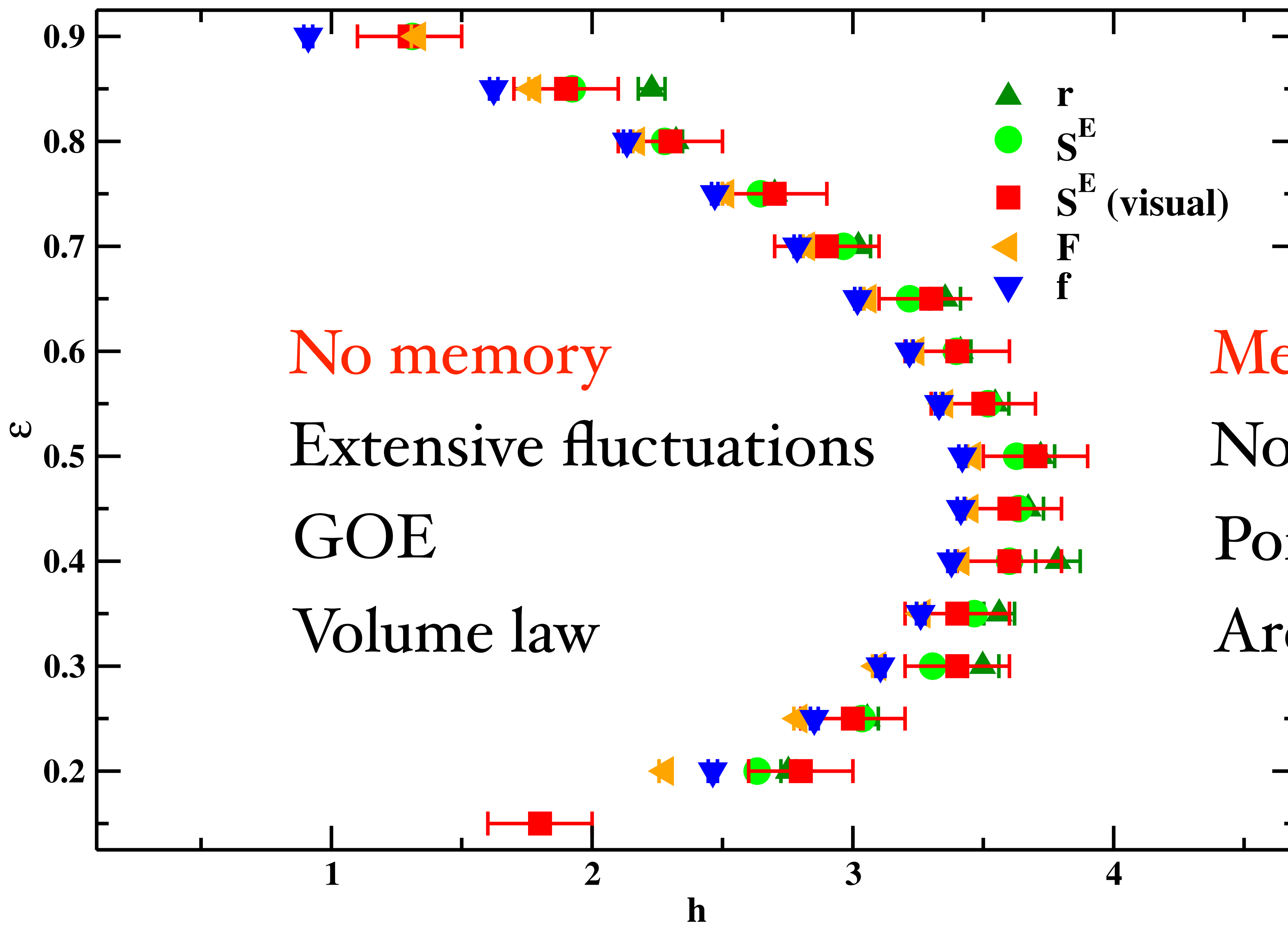


$f = 0$
«Memory»

$f = f_f(|h - h_c|.L^{1/\nu})$

Phase diagram (4)

- Energy-resolved phase diagram



No memory

Extensive fluctuations

GOE

Volume law

Memory

No fluctuations

Poisson

Area law

Localization in Hilbert space ?

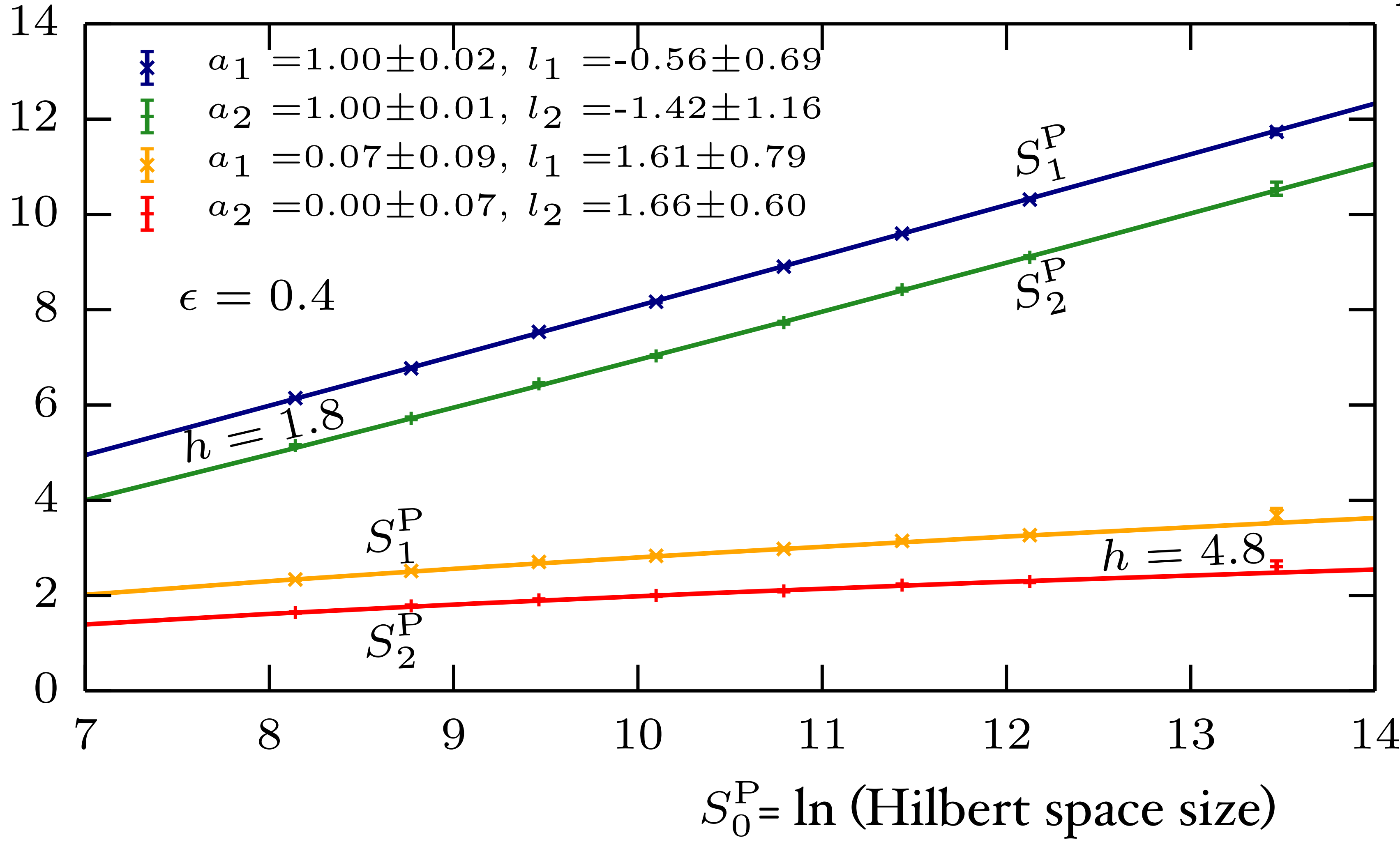
- How much a many-body wave-function is localized in a given basis?

$$|n\rangle = \sum_i n_i |i\rangle \qquad p_i = |\langle n|i\rangle|^2 \qquad \{|i\rangle\} = \{S^z\} \text{ basis}$$

Participation entropies $\quad S_1^p = -\sum_i p_i \ln(p_i) \qquad S_q^p = \frac{1}{1-q} \ln \sum_i p_i^q$

= ln (IPR)

- Scaling of participation entropy



Thermal:

$$S_q^p = a_q S_0^p - \ldots$$

$$a_q = 1$$

MBL: $\quad a_q \ll 1$

or

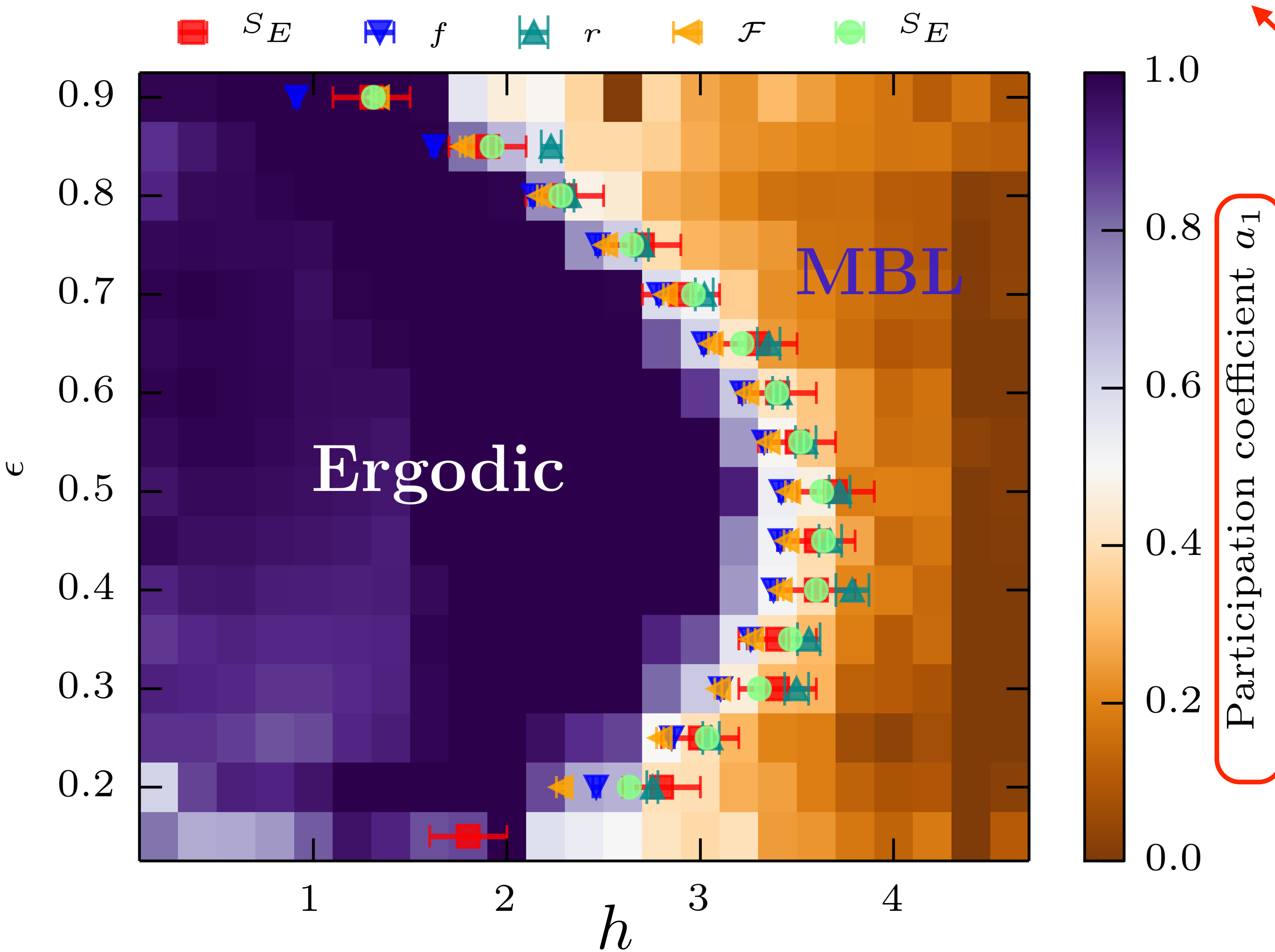
$$S_q^p = l_q \ln(S_o^p) + \ldots$$

No true localization in Hilbert space

Final phase diagram

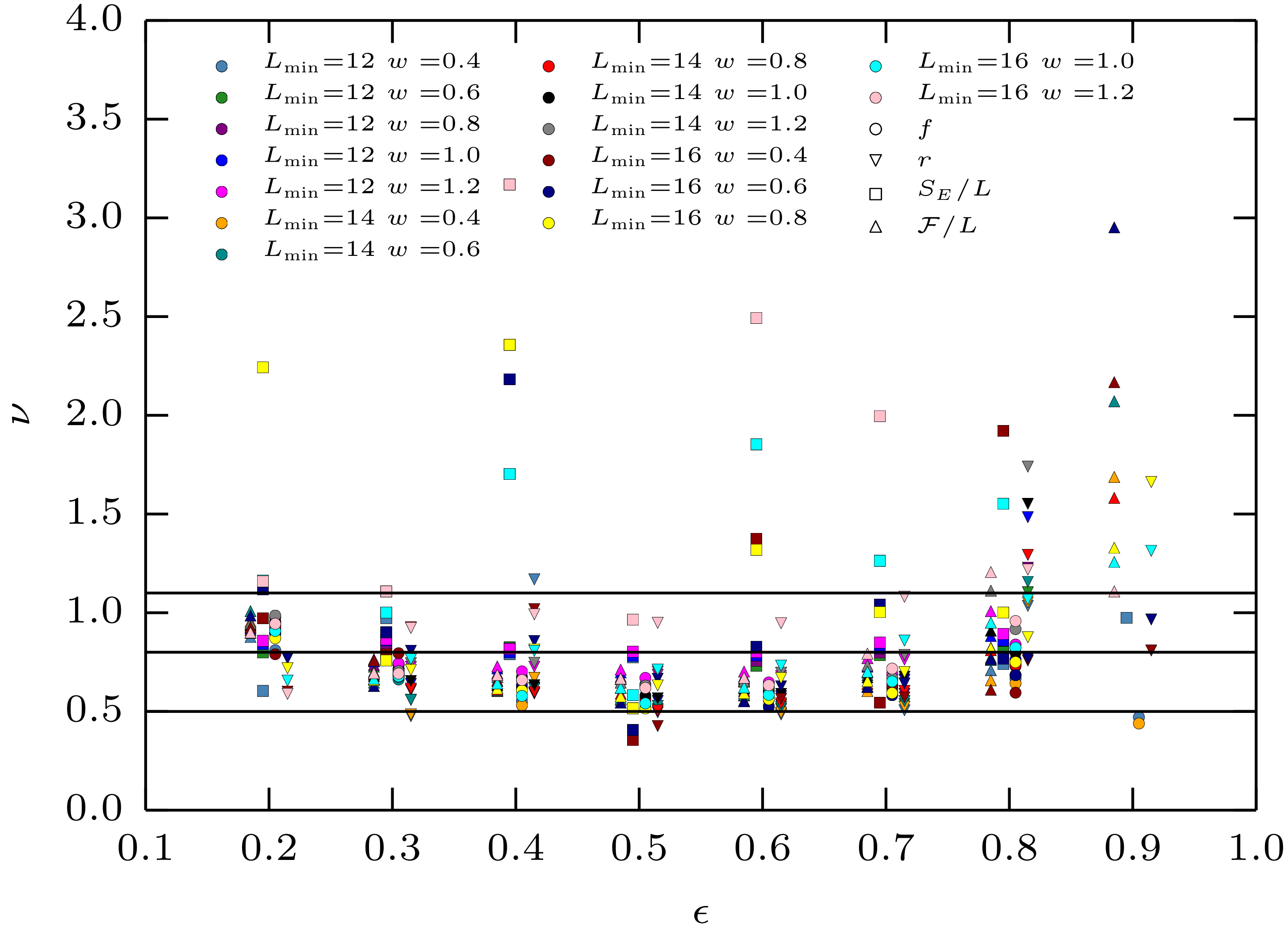
- Evidence for a many-body localization edge

$$S_1^p = a_1 S_0^p + \ldots$$



Critical exponent ?

- Systematic study of fit qualities to finite-size scaling ansätze
  - Different quantities
  - Starting from minimal size $L_{\min}$
  - Different fit windows
  - Including or not corrections to scaling ...



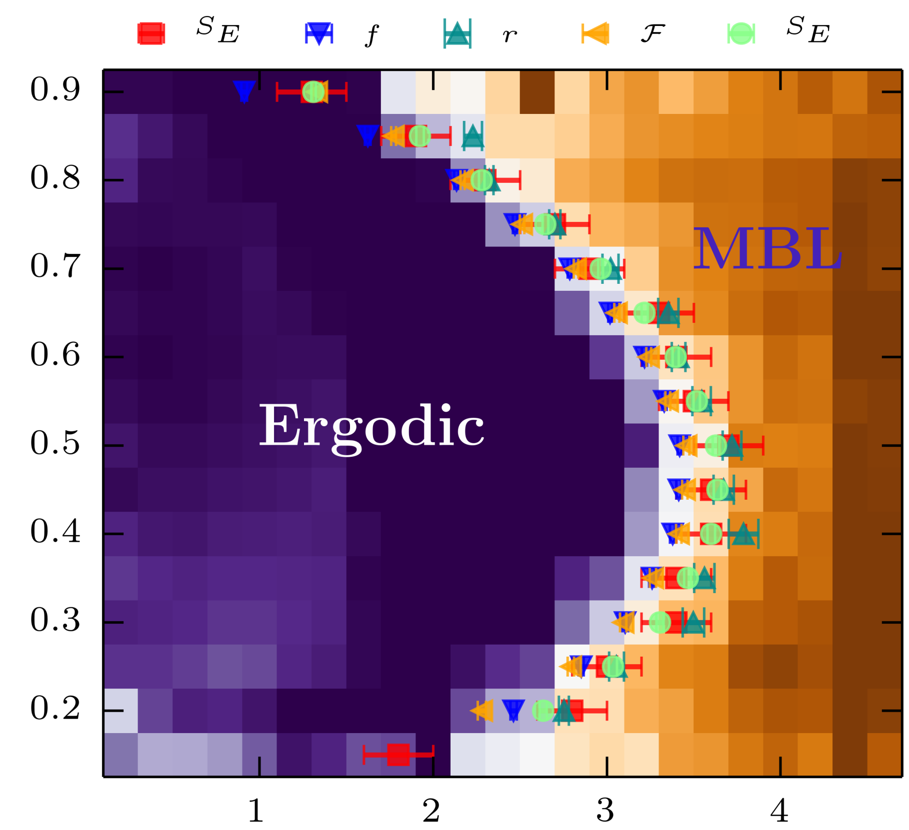
$\nu = 0.8(3)$

- Violation of Harris criterion $\nu \geq 2/d$??
  - Wrong form of ansätze ??
  - $L = 22$ too small ??
  - Rare events ??
- Effective models find $\nu \simeq 3$

Vosk *et al.*, Potter *et al.*

Conclusions & outlooks

- Based on improved numerics, evidence for:
  - Finite-size signatures of many-body localization
  - Presence of a many-body localization edge
  - No true localization in Hilbert space
  - Apparent violation of Harris criterion (?) within our system sizes



- Message 1: MBL is an active interesting field! Revisits usual stat-mech, connections to different fields (quantum chaos, information...)
- Message 2 : MBL is a computational challenge
- Message 3 : Many open questions

MBL in translation-invariant systems?

Many-body edge: Griffiths effects? sub-diffusive ergodic phase?

Nature of the transition?
New type of fixed point?

Experiments?