The semiclassical approximation in non-simply connected spaces

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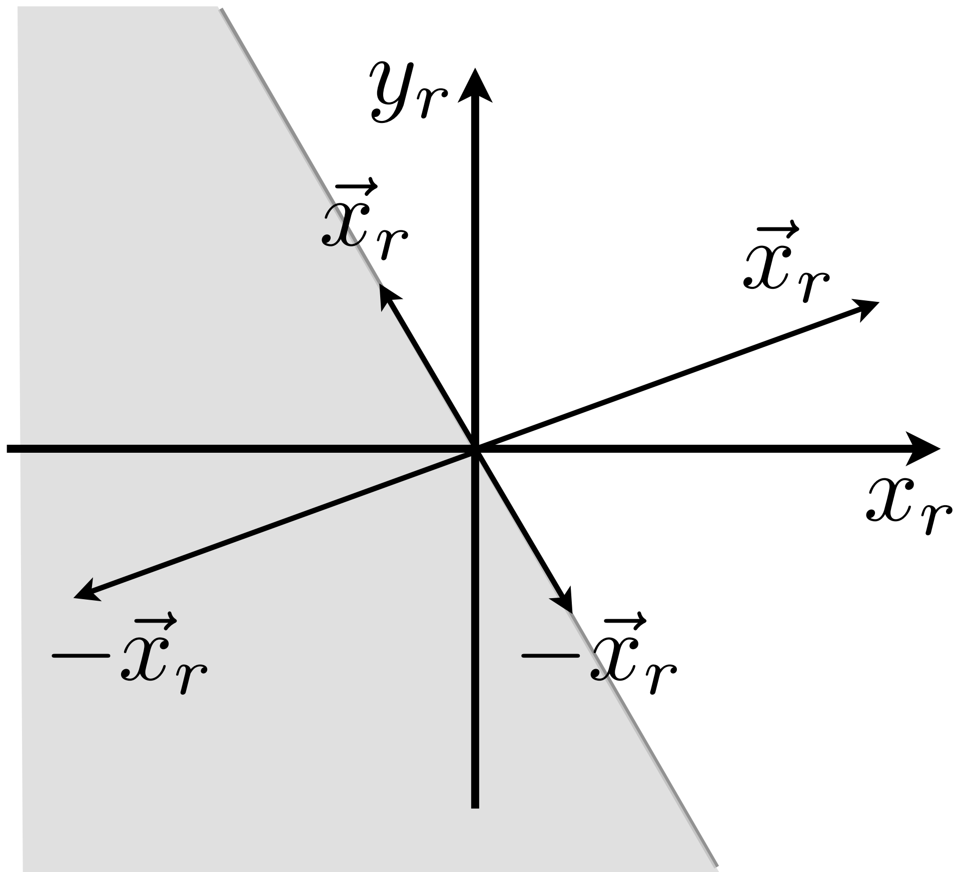
Non-simply connected spaces

Two identical particles in two dimensions

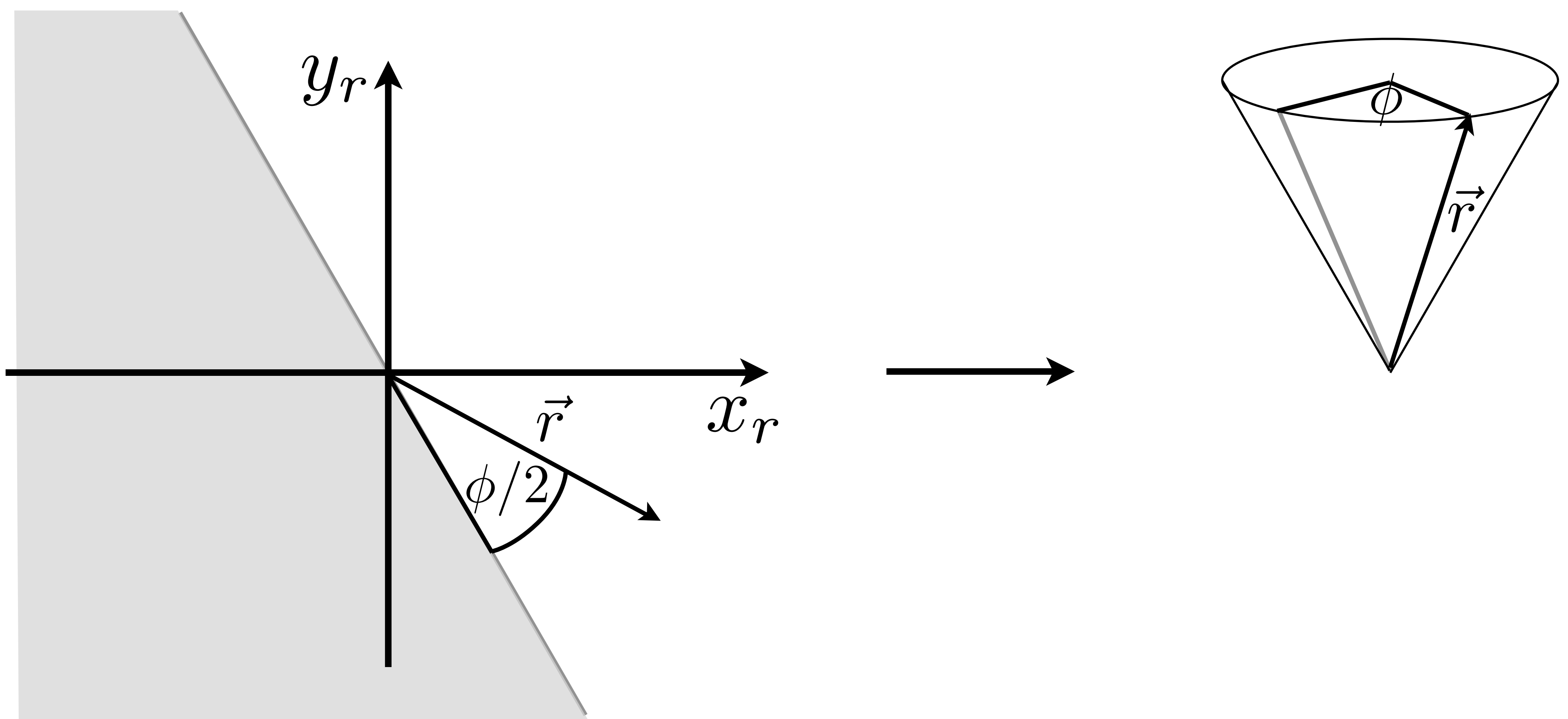


$$\vec{x}_{cm} = \frac{\vec{x}_1 + \vec{x}_2}{2}$$

$$\vec{x}_r = \vec{x}_1 - \vec{x}_2$$

$$S_{12}: \quad \vec{x}_1 \longleftrightarrow \vec{x}_2 \iff \begin{cases} \vec{x}_{cm} \longleftrightarrow \vec{x}_{cm} \\ \vec{x}_r \longleftrightarrow -\vec{x}_r \end{cases}$$

Two identical particles in two dimensions



J.M. Leinaas and J. Myrheim, Nuovo Cimento B **37**, 1-23 (1977).

Quantum mechanics

Quantum propagator

$$\psi\left(r,\phi+2\pi\right)=\exp\left(i\xi\right)\psi\left(r,\phi\right)$$

ξ odd multiple of π : fermions

ξ even multiple of π : bosons

$$\tilde{K}\left(r'',\phi'',r',\phi';t''-t'\right)$$
$$=\frac{m}{\pi i\hbar\left(t''-t'\right)}\exp\left(\frac{i}{\hbar}\frac{m}{2}\frac{r''^2+r'^2}{\left(t''-t'\right)}\right)\sum_{l=-\infty}^{\infty}I_{|2l+\xi/\pi|}\left(-\frac{i}{\hbar}\frac{mr''r'}{\left(t''-t'\right)}\right)e^{il\left(\phi''-\phi'\right)}$$

$$K_{b/f}\left(\vec{x}_r'',\vec{x}_r',t''-t'\right)$$
$$=\frac{m}{2\pi i\hbar\left(t''-t'\right)}\left(\exp\left(\frac{i}{\hbar}\frac{m}{2}\frac{\left(\vec{x}_r''-\vec{x}_r'\right)^2}{t''-t'}\right)\pm\exp\left(\frac{i}{\hbar}\frac{m}{2}\frac{\left(\vec{x}_r''+\vec{x}_r'\right)^2}{t''-t'}\right)\right)$$

Semiclassical dynamics

Classical functions

$$\tilde{H}\left(r,\phi,\frac{\partial}{\partial r},\frac{\partial}{\partial\phi}\right) = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{4}{r^2}\left(\frac{\partial}{\partial\phi}+i\frac{\xi}{2\pi}\right)^2\right)$$

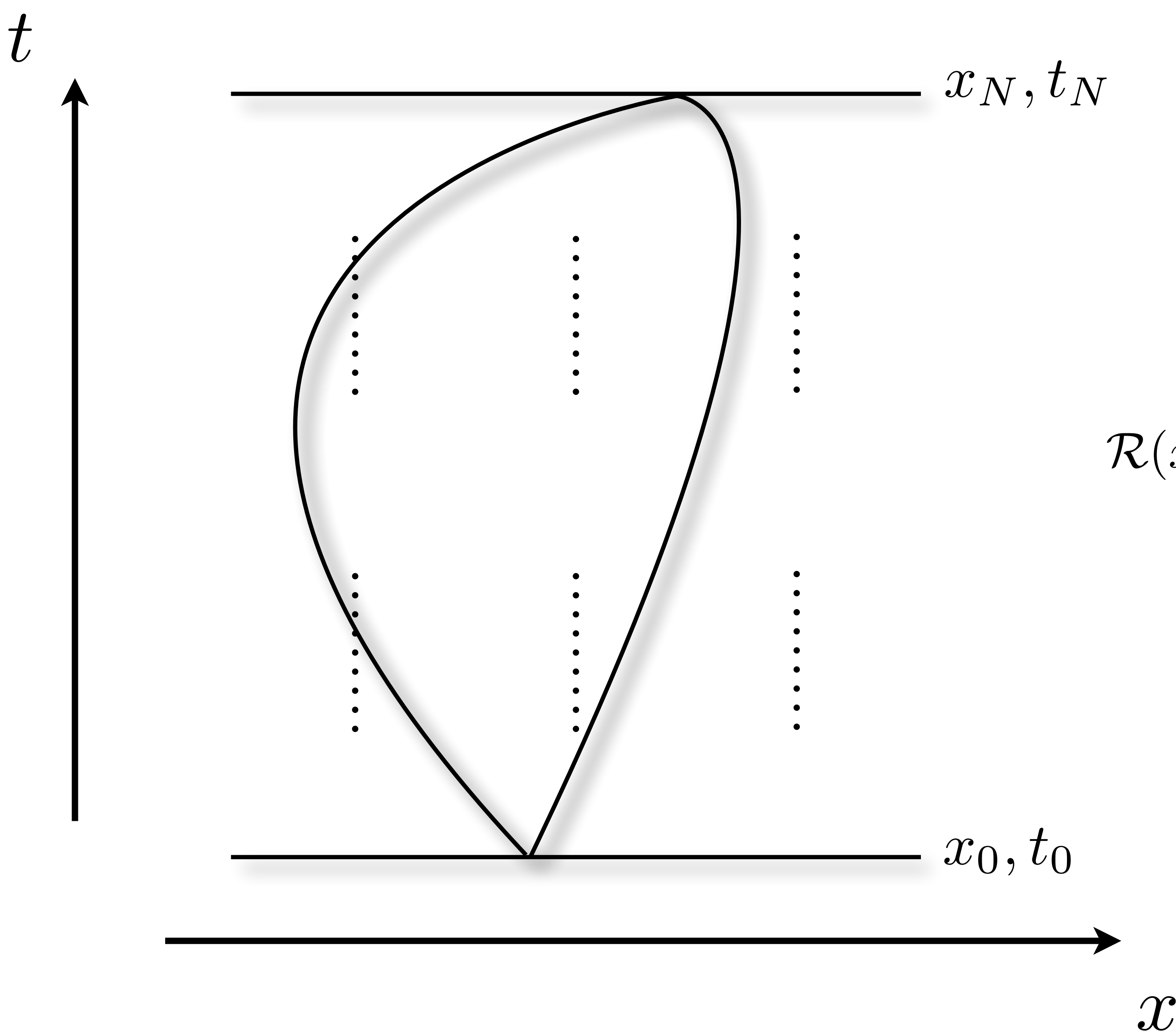
$$= -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)+\frac{1}{2m}\frac{4}{r^2}\left(\hat{p}_\phi+\frac{\Xi}{2\pi}\right)^2$$

$$\Xi = \hbar\xi$$

$$\mathcal{H} = \frac{1}{2m}\left(p_r^2+\frac{4}{r^2}\left(p_\phi+\frac{\Xi}{2\pi}\right)^2\right) \longrightarrow \mathcal{L} = \frac{1}{2}m\dot{r}^2+\frac{1}{8}mr^2\dot{\phi}^2-\frac{\Xi}{2\pi}\dot{\phi}$$

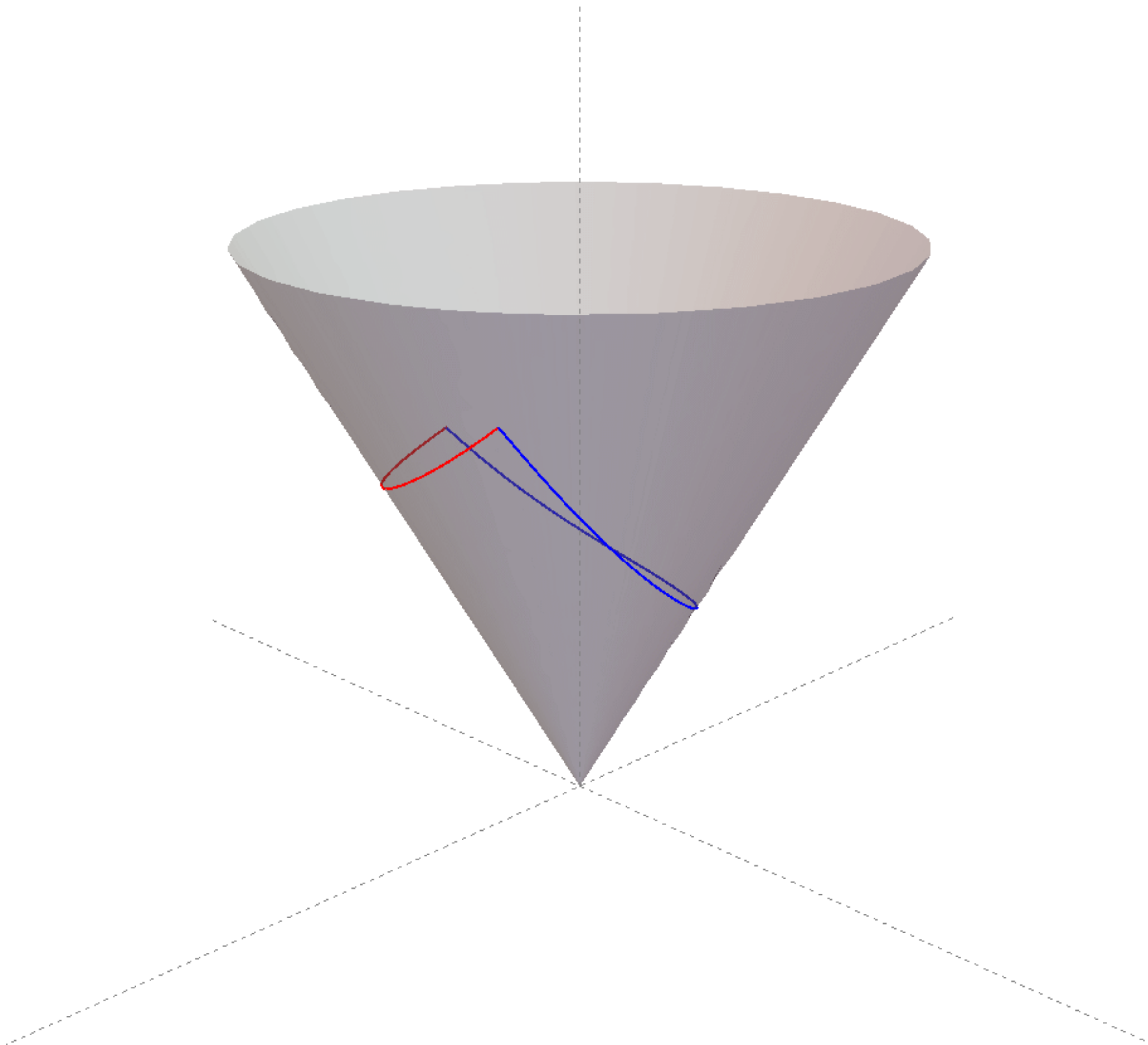
Path integrals and semiclassical physics

$$\mathcal{K}\left(\vec{x}'', \vec{x}'; t'', t'\right) = \frac{1}{\sqrt{2\pi i\hbar}^{f}} \sum_{\vec{x}_{cl.}} \sqrt{\left|\det\left(-\frac{\partial^2 \mathcal{R}(\vec{x}'', \vec{x}'; t'', t')}{\partial \vec{x}' \partial \vec{x}''}\right)\right|} \exp\left(\frac{i}{\hbar}\mathcal{R}(\vec{x}'', \vec{x}'; t'', t') - i\kappa\frac{\pi}{2}\right)$$

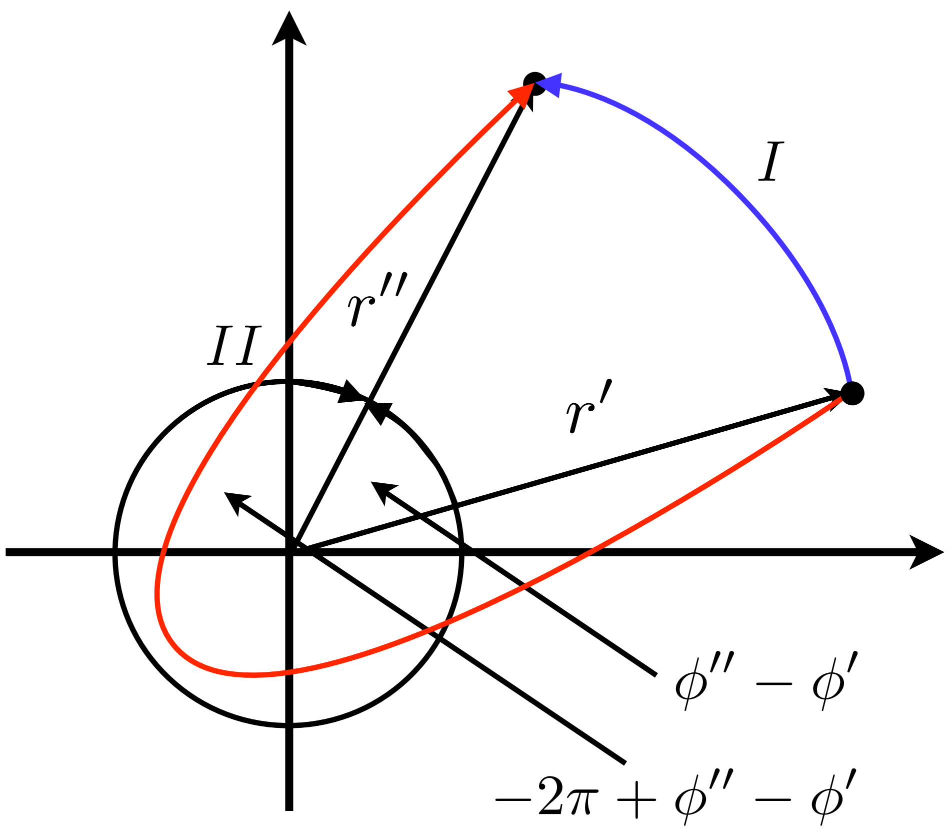


$$\mathcal{R}(\vec{x}'', \vec{x}'; t'', t') = \int_{t'}^{t''} \mathcal{L}\left(\vec{x}(t), \dot{\vec{x}}(t), t\right) dt$$

M.C. Gutzwiller, *The semi-classical quantization of chaotic Hamiltonian systems* in *Les Houches, Session LII, Chaos and Quantum Physics* (Elsevier, 1991)



Semiclassical propagator



$$\mathcal{L} = \frac{1}{2} m \dot{r}^2 + \frac{1}{8} m r^2 \dot{\phi}^2 - \frac{\Xi}{2\pi} \dot{\phi}$$

$$\frac{d}{dt}\left(m r^2 \dot{\phi}/4 - \frac{\Xi}{2\pi} \right) = 0$$

trajectories are not altered!

$$\mathcal{R} = \int_{t'}^{t''} \mathcal{L}\, dt = \mathcal{R}_0 - \frac{\Xi}{2\pi} \int_{t'}^{t''} \dot{\phi}\, dt$$

Semiclassical propagator

phase factor due to gauge transformation

extra phase accumulated by trajectory I

$$\widetilde{\mathcal{K}}(r'',\phi'',r',\phi';t'',t')$$

$$= \frac{m}{2\pi i\hbar\,(t''-t')} \exp\left(\frac{i\xi}{2\pi}(\phi''-\phi')\right)\left(\exp\left(\frac{i}{\hbar}\frac{m}{2}\frac{r''^2+r'^2-2r''r'\cos\left(\frac{\phi''-\phi'}{2}\right)}{t''-t'}-\frac{i\xi}{2\pi}(\phi''-\phi')\right)\right.$$

$$\left.+\exp\left(\frac{i}{\hbar}\frac{m}{2}\frac{r''^2+r'^2+2r''r'\cos\left(\frac{\phi''-\phi'}{2}\right)}{t''-t'}-\frac{i\xi}{2\pi}(\phi''-\phi'\mp 2\pi)\right)\right)$$

extra phase accumulated by trajectory II

$$\mathcal{K}\left(\vec{x}_r'',\vec{x}_r';t''-t'\right)=\frac{m}{2\pi i\hbar\,(t''-t')}\left(\exp\left(\frac{i}{\hbar}\frac{m}{2}\frac{\left(\vec{x}_r''-\vec{x}_r'\right)^2}{t''-t'}\right)+e^{\pm i\xi}\exp\left(\frac{i}{\hbar}\frac{m}{2}\frac{\left(\vec{x}_r''+\vec{x}_r'\right)^2}{t''-t'}\right)\right)$$

$$\text{where}\quad\begin{cases} ''+'' & \text{for} \quad p_\phi>0\\ ''-'' & \text{for} \quad p_\phi<0\end{cases}$$

Synopsis

Q.M.

$$K\left(r'',\phi'',r',\phi';t''-t'\right)=\frac{m}{\pi i\hbar\left(t''-t'\right)}\exp\left(\frac{i}{\hbar}\frac{m}{2}\frac{r''^2+r'^2}{\left(t''-t'\right)}\right)\sum_{l=-\infty}^{\infty}I_{|2l+\xi/\pi|}\left(-\frac{i}{\hbar}\frac{mr''r'}{\left(t''-t'\right)}\right)e^{i(l+\xi/2\pi)\left(\phi''-\phi'\right)}$$

S.C.

$$\mathcal{K}\left(\vec{x}''_r,\vec{x}'_r;t''-t'\right)=\frac{m}{2\pi i\hbar\left(t''-t'\right)}\left(\exp\left(\frac{i}{\hbar}\frac{m}{2}\frac{\left(\vec{x}''_r-\vec{x}'_r\right)^2}{t''-t'}\right)+\mathrm{e}^{\pm i\xi}\exp\left(\frac{i}{\hbar}\frac{m}{2}\frac{\left(\vec{x}''_r+\vec{x}'_r\right)^2}{t''-t'}\right)\right)$$

$$\text{where}\quad\begin{cases}\text{"}+\text{"} & \text{for}\quad p_\phi>0\\ \text{"}-\text{"} & \text{for}\quad p_\phi<0\end{cases}$$

Time-sliced propagator

$$\tilde{K}\left(r_n,\phi_n,r_{n-1},\phi_{n-1};\tau\right)$$

$$\sim \frac{m}{2\pi i\hbar\tau}\exp\left(\frac{i}{\hbar}\left(\frac{m}{2}\left(\frac{r_n-r_{n-1}}{\tau}\right)^2+\frac{m}{8}r_nr_{n-1}\left(\frac{\phi_n-\phi_{n-1}}{\tau}\right)^2-\frac{\Xi}{2\pi}\frac{\phi_n-\phi_{n-1}}{\tau}\right)\tau\right)$$

$$\times\sqrt{\frac{i\tau}{\hbar}\frac{\lambda}{\pi}}2\int_{-\infty}^{\infty}dp_\phi\,\exp\left(-\frac{i}{\hbar}\lambda 4p_\phi^2\tau\right)$$

$$+\frac{m}{2\pi i\hbar\tau}\exp\left(\frac{i}{\hbar}\left(\frac{m}{2}\left(\frac{r_n+r_{n-1}}{\tau}\right)^2-\frac{m}{8}r_nr_{n-1}\left(\frac{\phi_n-\phi_{n-1}}{\tau}\right)^2-\frac{\Xi}{2\pi}\frac{\phi_n-\phi_{n-1}}{\tau}\right)\tau\right)$$

$$\times\left(\exp\left(-\frac{i}{\hbar}\frac{\Xi}{2\pi}\frac{2\pi}{\tau}\tau\right)\sqrt{\frac{\tau}{i\hbar}\frac{\lambda}{\pi}}2\int_{p_\phi^{cl.}}^{\infty}dp_\phi\,\exp\left(\frac{i}{\hbar}\lambda 4p_\phi^2\tau\right)\right.$$

$$\left.+\exp\left(\frac{i}{\hbar}\frac{\Xi}{2\pi}\frac{2\pi}{\tau}\tau\right)\sqrt{\frac{\tau}{i\hbar}\frac{\lambda}{\pi}}2\int_{\infty}^{p_\phi^{cl.}}dp_\phi\,\exp\left(\frac{i}{\hbar}\lambda 4p_\phi^2\tau\right)\right)$$

$$p_\phi^{cl.}=mr_nr_{n-1}\left(\phi_n-\phi_{n-1}\right)/4\tau$$

Time-sliced propagator

$$\tilde{K}\left(r_n, \phi_n, r_{n-1}, \phi_{n-1}; \tau\right)$$

$$\sim \frac{m}{2\pi i\hbar\tau} \exp\left(\frac{i}{\hbar}\left(\frac{m}{2}\left(\frac{r_n - r_{n-1}}{\tau}\right)^2 + \frac{m}{8} r_n r_{n-1}\left(\frac{\phi_n - \phi_{n-1}}{\tau}\right)^2 - \frac{\Xi}{2\pi}\frac{\phi_n - \phi_{n-1}}{\tau}\right)\tau\right)$$

$$+ \frac{m}{2\pi i\hbar\tau} \exp\left(\frac{i}{\hbar}\left(\frac{m}{2}\left(\frac{r_n + r_{n-1}}{\tau}\right)^2 - \frac{m}{8} r_n r_{n-1}\left(\frac{\phi_n - \phi_{n-1}}{\tau}\right)^2 - \frac{\Xi}{2\pi}\frac{\phi_n - \phi_{n-1} \mp 2\pi}{\tau}\right)\tau\right)$$

$$\text{where} \quad \begin{cases} " + " & \text{for} \quad p_\phi^{cl.} > 0 \\ " - " & \text{for} \quad p_\phi^{cl.} < 0 \end{cases}$$

$$\tilde{\mathcal{K}}(r'', \phi'', r', \phi'; t'' - t')$$

$$= \frac{m}{2\pi i\hbar\left(t'' - t'\right)}\left(\exp\left(\frac{i}{\hbar}\left(\frac{m}{2}\frac{r''^2 + r'^2 - 2r''r'\cos\left(\frac{\phi''-\phi'}{2}\right)}{t'' - t'} - \frac{\Xi}{2\pi}\left(\phi'' - \phi'\right)\right)\right)\right.$$

$$\left. + \exp\left(\frac{i}{\hbar}\left(\frac{m}{2}\frac{r''^2 + r'^2 + 2r''r'\cos\left(\frac{\phi''-\phi'}{2}\right)}{t'' - t'} - \frac{\Xi}{2\pi}\left(\phi'' - \phi' \mp 2\pi\right)\right)\right)\right)$$

$$\text{where} \quad \begin{cases} " + " & \text{for} \quad p_\phi > 0 \\ " - " & \text{for} \quad p_\phi < 0 \end{cases}$$

Conclusions

- new method to obtain certain time-sliced semiclassical propagators
- quantized momenta are turned into their classical analogues, Bessel functions into exponentials
- derivation applicable to problems in infinitely connected spaces (anyons, Aharonov-Bohm potential), spherical polar coordinates…