The semiclassical method in interacting many body systems

Path integrals, fields and particles

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Quantum chaos: methods and applications, March 2015

Motivation: The subject (fields)

Subject:

- Quantum Fields with large number of excitations ($N \to \infty$)
- So far:
  $\to$ Finite (Trying to do scattering)
  $\to$ Lattice (Trying to get continuum)
  $\to$ Non-Relativistic (Dreaming about QCD)
  $\to$ Isolated (Trying to do Feynman-Vernon)
- interactions
  $\to$ General

Motivation: The goal (fields)

Goal: Study analytically non-perturbative effects in $\hbar_{\text{eff}} = 1/N$

- Use only classical info (actions, stabilities, etc) from a classical (nonlinear) field equation

Which properties?:

- dynamics (a van Vleck -Gutzwiller propagator)
- spectrum (a Gutzwiller trace formula)
- Thermodynamics

How?:

- For the moment, using universality:
  $\rightarrow$ Due to single-particle chaos (Mesoscopic Boson Sampling)
  $\rightarrow$ Due to field chaos (Thomas Engl, Peter Schlagheck)

Break: summarizing fields

Semiclassical propagator a la Gutzwiller:

- Write $K(\psi^{\text{in}}, \psi^{\text{fin}}, t) = \int \mathcal{D}[\psi(s)] \mathrm{e}^{iR[\psi(s)]/\hbar_{\text{eff}}}$
- Define classical limit $\delta_\psi R[\psi(s)] = 0$ and B.C.
- Evaluate in Stationary Phase Approximation
- Careful:
  $\rightarrow$ Coherent states (Bosons, Fermions,Klauder) $\rightarrow$ bad classical limit
  $\rightarrow$ Extra conditions (large densites, gauge symmetries)
- van Vleck-Gutzwiller propagator (Bosons, Fermions)
- Gutzwiller trace formula (Bosons)

Motivation: The subject (particles)

Subject:

- fixed number of identical particles ($N \sim \mathcal{O}(10)$)
- simple external potentials:
  $\rightarrow$ free quantum gases with periodic boundary conditions
  $\rightarrow$ quantum billiards
  $\rightarrow$ harmonic traps
  $\rightarrow$ other homogeneous potentials
- isolated or in a thermal bath
- interactions between particles
  $\rightarrow$ model: contact-interaction

Motivation: The goal (particles)

Goal: Study properties related to MB-spectrum analytically and non-perturbatively

- avoid numerical calculation of MB-energy levels and power series in interaction strength

Which properties?:

- the spectrum itself
- canonical partition function
- equation of state
- spatial properties:
  $\rightarrow$ (non-local) pair correlations in (micro-)canonical ensemble
  $\rightarrow$ spatial particle densities near boundaries/impurities

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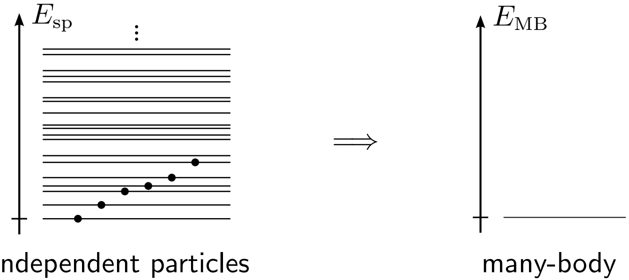
Many-body(MB)-spectrum: General consideration

- around ground state: mean-field approaches work
  $\rightarrow$ effectively independent particles (HF, . . .)



Many-body(MB)-spectrum: General consideration

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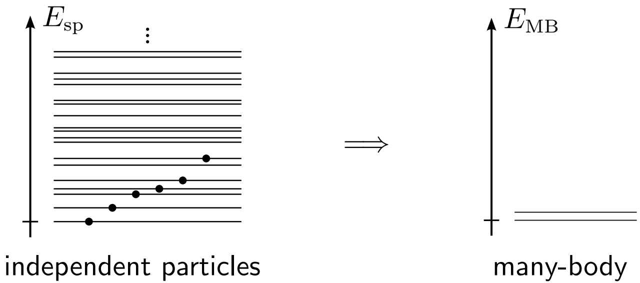

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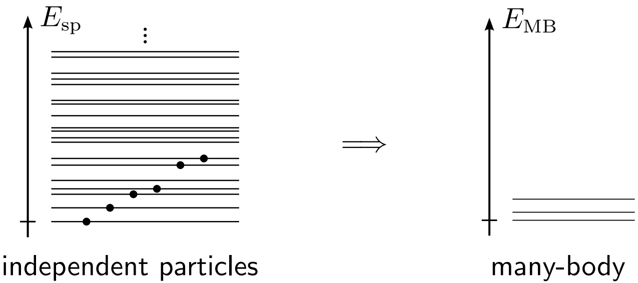

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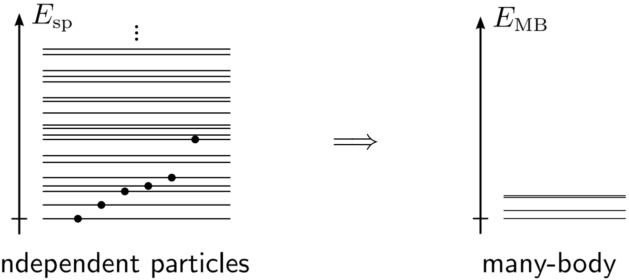

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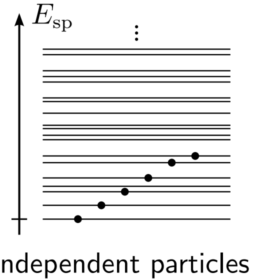
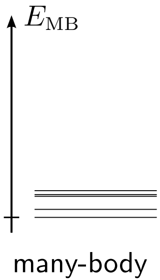


$\Longrightarrow$



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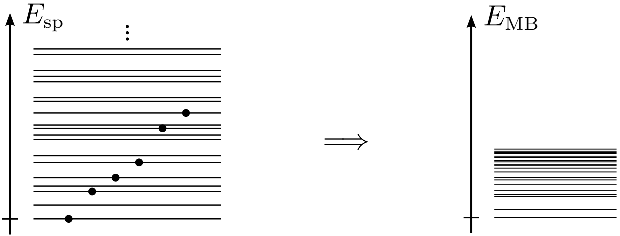

independent particles many-body

Many-body(MB)-spectrum: General consideration

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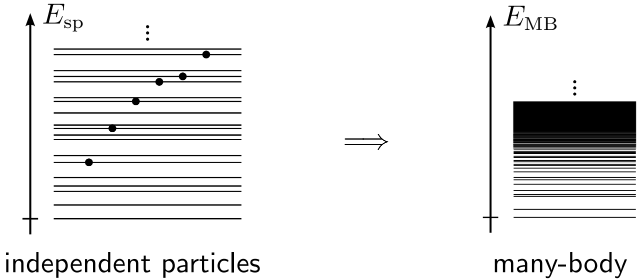
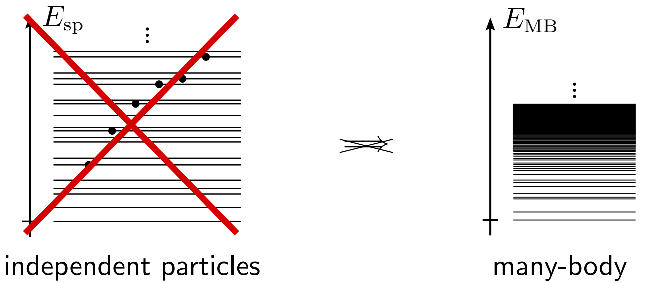


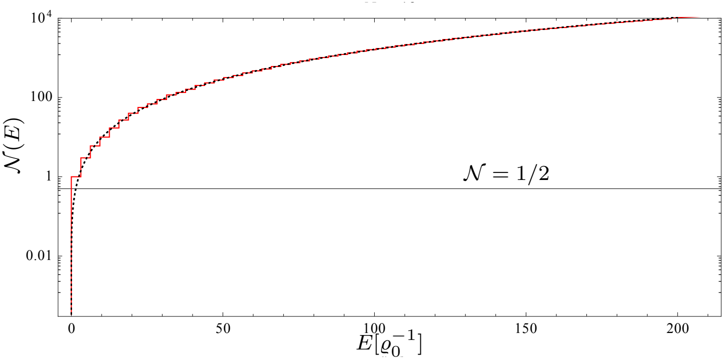
Many-body(MB)-spectrum: General consideration

- around ground state: mean-field approaches work
  $\rightarrow$ effectively independent particles (HF, ...)
- **but:** no single mean field for all excitations! (MCSCF, ...)



Simplification

Simplification: Forget about discreteness of spectrum!

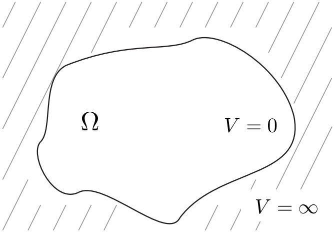


Overview

Single particle(sp) Weyl expansion

1 particle, D-dim. billiard

$$\varrho_{\mathrm{sp}}(E) = \mathcal{FT}_t\left[\int \mathrm{d}q K(\mathbf{q},\mathbf{q};t)\right](E)$$



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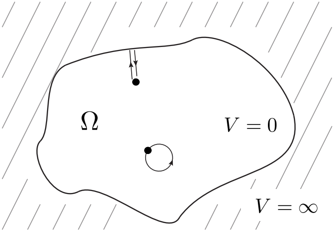


smooth part $\bar{\varrho}$
$\leftrightarrow$ short time behaviour of K

$$\bar{\varrho}_{\mathrm{sp}}(E) = \underbrace{\text{const.} \cdot V_D \, E^{\frac{D}{2}-1}}_{\text{locally free}} - \underbrace{\text{const.} \cdot S_{D-1} \, E^{\frac{D-1}{2}-1}}_{\text{reflection on flat boundary}} + \ldots$$

$\rightarrow$ basic geometric properties!

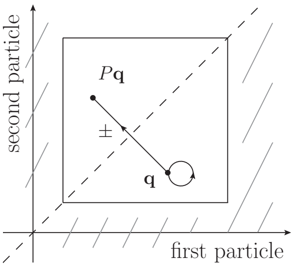
MB Weyl expansion for bosons (+) or fermions (−)

N identical particles $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$

$$\varrho_{\pm}^{(N)}(E) = \mathcal{FT}_t \left[\int \mathrm{d}q \, K_{\pm}^{(N)}(\mathbf{q}, \mathbf{q}; t) \right] (E)$$

$$K_{\pm}^{(N)}(\mathbf{q}, \mathbf{q}; t) = \frac{1}{N!} \sum_P (\pm 1)^P K^{(N)}(P\mathbf{q}, \mathbf{q}; t)$$

2 particles, 1D



smooth part $\bar{\varrho}_{\pm}$, **non-interacting**:

Q. Hummel, J. D. Urbina and K. Richter, J. Phys. A: Math. Theor. **47**, 015101 (2014):

P-contributions ↔ Weyl-like corrections

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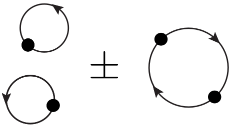

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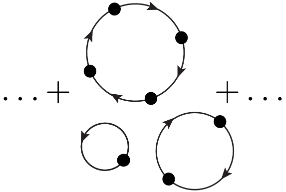
N particles



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MB Weyl expansion for non-int. bosons (+) or fermions (−)

$$\bar{\varrho}_{\pm}(E) = \sum_{l_1,l_2=1}^{N} C_{l_1,l_2}\, V_D^{l_1}\, S_{D-1}^{l_2}\, E^{l_1 D/2 + l_2 (D-1)/2 - 1}$$

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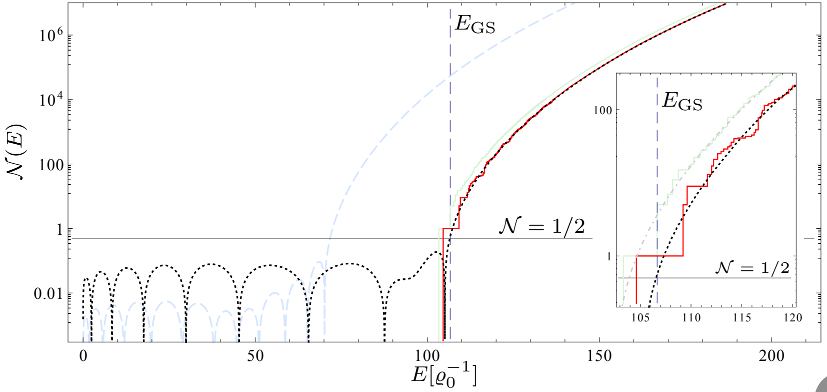
12 fermions, 2D, counting function $\mathcal{N}(E) = \int \mathrm{d}E\, \bar{\varrho}(E)$



MB Weyl expansion for non-int. bosons (+) or fermions (−)

$$\bar{\varrho}_{+}(E) = \sum_{l=1}^{N} C_l \, L^l \, E^{l/2-1}$$

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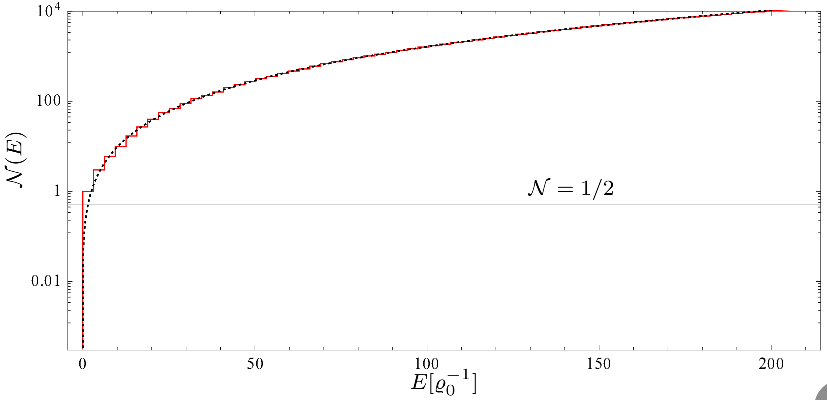
7 bosons, 1D, counting function $\mathcal{N}(E) = \int \mathrm{d}E \, \bar{\varrho}(E)$



Thermodynamics: Switching between domains

Relation between E, t, β domains:

energy domain E: $\bar{\varrho}(E)$
smooth part of DOS

time domain t: $\mathrm{tr}\, K(t) = \int \mathrm{d}E \bar{\varrho}(E) \mathrm{e}^{-\frac{i}{\hbar}Et}$
short time propagation

inv. temperature β: $Z(\beta) = \mathcal{L}_{\mathrm{E}}[\bar{\varrho}(E)](\beta)$
(imaginary time $\beta = \frac{i}{\hbar}t$)
"high" temperature behaviour of canonical partition function
$V/\lambda_T^D \gtrsim N \,, \quad \lambda_T = \sqrt{4\pi\beta}$

Thermodynamics: Canonical vs. grand canonical

Ideal quantum gas in D-dim.

canonical:
(our approach)

$$Z(\beta) = \sum_{l=1}^{N} z_l \left(\frac{V}{\lambda_T^D}\right)^l, \quad \lambda_T = \sqrt{4\pi\beta}$$

"high" temperature ↔ no discreteness

grand canonical:
(std. textbook)

$$\begin{aligned} \ln Z_\mathrm{G} &= \mp \sum_i \ln(1 \mp \mathrm{e}^{-\beta(\epsilon_i - \mu)}) \\ &\approx \mp \int \mathrm{d}\epsilon\, \bar{\varrho}_\mathrm{sp}(\epsilon) \ln(1 \mp \mathrm{e}^{-\beta(\epsilon-\mu)}) \\ &= \pm \frac{V}{\lambda_T^D} \mathrm{Li}_{\frac{D}{2}+1}(\pm \mathrm{e}^{\beta\mu}), \quad \mathrm{Li}_s(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^s} \end{aligned}$$

no discreteness as well!

→ "equal footing", different ensembles

Thermodynamics: Canonical vs. grand canonical

Equation of state (EOS) for a given N

canonical:
(our approach)

$$P(V,T,N) = \frac{k_\mathrm{B}T}{V} \frac{\sum_{l=1}^{N} z_l l \left(\frac{V}{\lambda_T^D}\right)^l}{\sum_{l=1}^{N} z_l \left(\frac{V}{\lambda_T^D}\right)^l}$$

grand canonical:
(std. textbook)

$$P(V,T,\mu) = \pm \frac{k_\mathrm{B}T}{\lambda_T^D} \mathrm{Li}_{\frac{D}{2}+1}(\pm \mathrm{e}^{\beta\mu})$$

$$< N >= \pm \frac{V}{\lambda_T^D} \mathrm{Li}_{\frac{D}{2}}(\pm \mathrm{e}^{\beta\mu})$$

Thermodynamics: Canonical vs. grand canonical

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grand canonical:
(std. textbook)

$$P(V,T,\mu) \sim c_1 z + c_2 z^2 + c_3 z^3 + \cdots\ , \quad z = \mathrm{e}^{\beta\mu}$$

$$< N > \sim c_1 z + 2c_2 z^2 + 3c_3 z^3 + \cdots$$

Thermodynamics: Canonical vs. grand canonical

Equation of state (EOS) for a given N

canonical:
(our approach)

$$P(V,T,N) = \frac{k_\mathrm{B}T}{V}\frac{\sum_{l=1}^{N} z_l l \left(\frac{V}{\lambda_T^D}\right)^l}{\sum_{l=1}^{N} z_l \left(\frac{V}{\lambda_T^D}\right)^l}$$

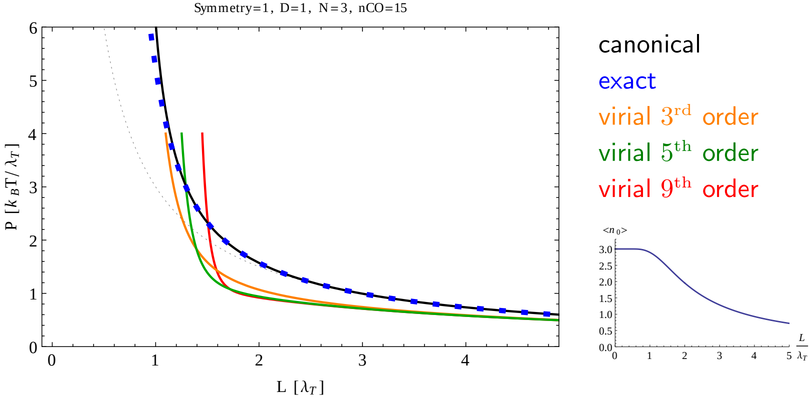
grand canonical:
(std. textbook)

$$P = nk_\mathrm{B}T(1 + a_2 n\lambda_T^D + a_3 n^2 \lambda_T^{2D} + \cdots)$$

$\rightarrow$ high temperature!
$\rightarrow$ convergence?

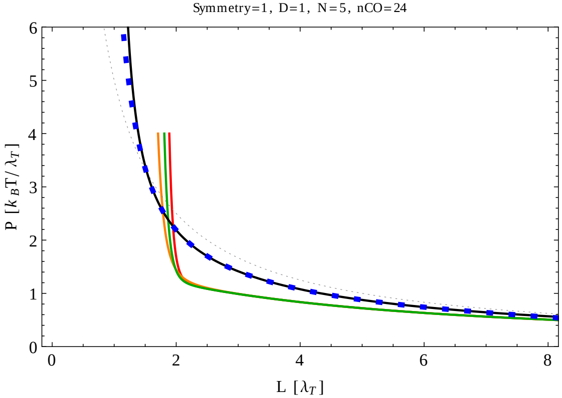
Thermodynamics: Canonical vs. grand canonical

Ideal Bose gas, 1D-box, EOS for $N = 3$

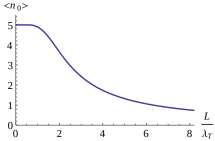


Thermodynamics: Canonical vs. grand canonical

Ideal Bose gas, 1D-box, EOS for $N = 5$

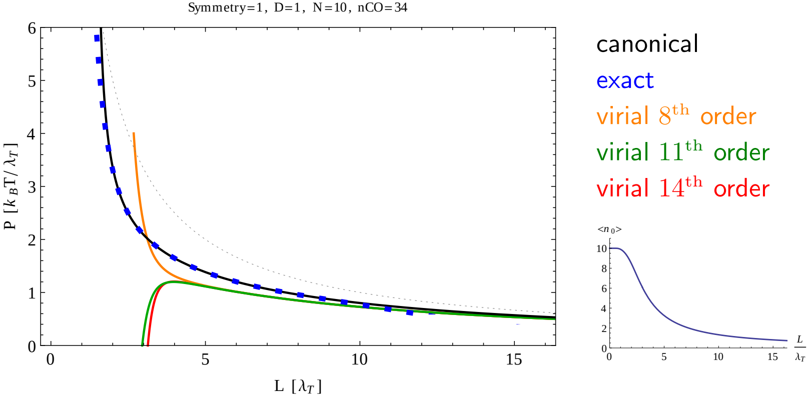


canonical
exact
virial 8^{th} order
virial 11^{th} order
virial 14^{th} order



Thermodynamics: Canonical vs. grand canonical

Ideal Bose gas, 1D-box, EOS for $N = 10$



Thermodynamics: Canonical vs. grand canonical

Fluctuations in Ideal and Interacting Bose-Einstein Condensates:
From the laser phase transition analogy
to squeezed states and Bogoliubov quasiparticles *

Vitaly V. Kocharovsky[1,2], Vladimir V. Kocharovsky[2], Martin Holthaus[3], C. H. Raymond Ooi[1], Anatoly Svidzinsky[1], Wolfgang Ketterle[4], and Marlan O. Scully[1,5]
[1]*Institute for Quantum Studies and Dept. of Physics, Texas A&M Univ.,TX 77843-4242*
[2]*Institute of Applied Physics, Russian Academy of Science, 600950 Nizhny Novgorod, Russia*
[3]*Institut für Physik, Carl von Ossietzky Universitat, D-2611 Oldenburg, Germany*
[4]*MIT-Harvard Center for Ultracold Atoms, and Dept. of Physics, MIT, Cambridge, Mass. 02139*
[5]*Princeton Institute for Materials Science and Technology, Princeton Univ., NJ 08544-1009*
(Dated: February 4, 2008)

We review the phenomenon of equilibrium fluctuations in the number of condensed atoms n_0 in a trap containing N atoms total. We start with a history of the Bose-Einstein distribution, a similar grand canonical problem with an indefinite total number of particles, the Einstein-Uhlenbeck debate concerning the rounding of the mean number of condensed atoms $\bar{n}_0$ near a critical temperature T_c, and a discussion of the relations between statistics of BEC fluctuations in the grand canonical,

"... In view of the present experimental status the canonical and microcanonical descriptions of the BEC are of primary importance. Recent BEC experiments on harmonically trapped atoms of dilute gases deal with a finite and well defined number of particles. ... "

V. V. Kocharovsky et al, Advances in Atomic, Molecular and Optical Physics **53**, 291 (2006)

Thermodynamics: Canonical vs. grand canonical

W. J. Mullin and J. P. Fernandez, Am. J. Phys. **71**, 661 (2003):

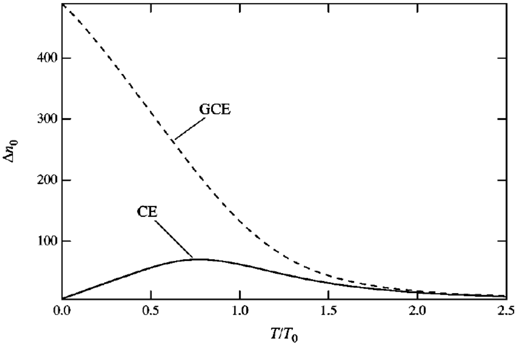


Fig. 4. Root-mean-square fluctuation of the number of particles in the ground state of the 1D harmonic Bose gas for the grand canonical and canonical ensembles. The fluctuations in the condensate in the grand canonical ensemble become as large as the occupation itself, which is unphysical. The canonical ensemble result is more reasonable.

Overview

Lieb-Liniger model

- N bosons in 1D, $x_i \in [0, L]$, periodic boundary conditions

$$\hat{H} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sqrt{8\alpha} \sum_{i<j} \delta(x_i - x_j)$$

- exactly solvable (Bethe ansatz): $E_{\mathbf{I}} = \sum_i k_i^2$

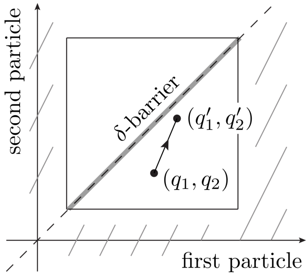
$$Lk_i = 2\pi I_i - 2\sum_{j=1}^{N} \arctan\left(\frac{k_i - k_j}{\sqrt{2\alpha}}\right), \quad i = 1, \ldots, N$$

where $I_1 < \cdots < I_N$ are $\begin{cases} \text{integer} & N \text{ odd} \\ \text{half-integer} & N \text{ even} \end{cases}$

- describes ultracold atoms

2 particles, 1D, contact-interaction

$$\hat{H} = -\left(\frac{\partial^2}{\partial q_1^2} + \frac{\partial^2}{\partial q_2^2}\right) + \sqrt{8\alpha}\,\delta(q_1 - q_2)\,, \quad \alpha > 0$$



smooth part ($t \to 0$, small β):
no reflections from boundary

$\Downarrow$

CM (R): free propagator K_0

rel (r): propagator K_δ of δ-potential placed in free space

2 particles, 1D, contact-interaction

distinguishable case:

$$K^{(2)}(\mathbf{q}', \mathbf{q}; t) = K_0^{(\mathrm{CM})}(R', R; t) \cdot K_\delta^{(\mathrm{rel})}(r', r; t)$$

2 particles, 1D, contact-interaction

distinguishable case:

$$K^{(2)}(\mathbf{q}', \mathbf{q}; t) = K_0^{(\mathrm{CM})}(R', R; t) \cdot K_\delta^{(\mathrm{rel})}(r', r; t)$$

with

$$K_\delta^{(\mathrm{rel})}(r', r; t) = K_0^{(\mathrm{rel})}(r', r; t) + K_\alpha^{(\mathrm{rel})}(r', r; t)\,,$$

where

$$K_\alpha^{(\mathrm{rel})}(r', r; t) = -\alpha^{\frac{1}{2}} \left(\frac{\hbar}{4\pi \mathrm{i} t}\right)^{\frac{1}{2}} \int_0^\infty \mathrm{d}u\, \mathrm{e}^{-\left(\frac{\alpha}{2}\right)^{\frac{1}{2}} u + \frac{\mathrm{i}\hbar}{8t}(|r'|+|r|+u)^2}$$

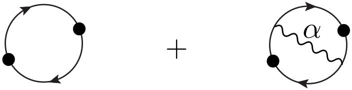
2 particles, 1D, contact-interaction

distinguishable: $K^{(2)} = K_0^{(2)} + K_0^{(\mathrm{CM})} K_\alpha^{(\mathrm{rel})}$

indistinguishable: $K_\pm^{(2)} = K_{0,\pm}^{(2)} + \frac{1}{2}(1 \pm 1) K_0^{(\mathrm{CM})} K_\alpha^{(\mathrm{rel})}$

$\mathrm{tr}\, K^{(2)}(\mathbf{q}, \mathbf{q}; t) \;\widehat{=}$

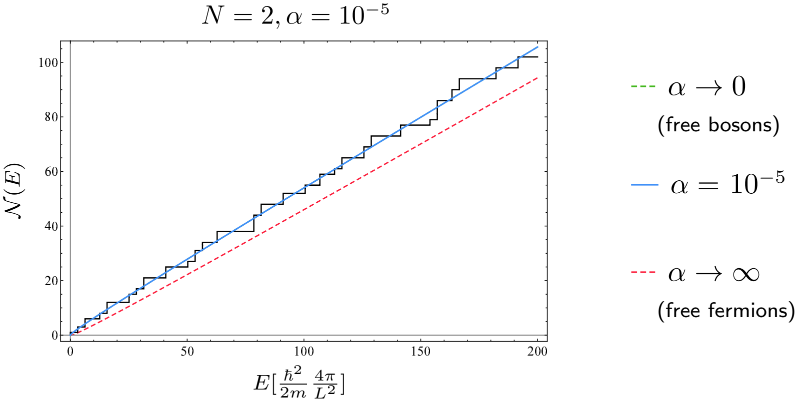
$\mathrm{tr}\, K^{(2)}(P_{12}\mathbf{q}, \mathbf{q}; t) \;\widehat{=}$



No Feynman diagrams!

2 bosons, counting function $\mathcal{N}(E) = \int \mathrm{d}E\, \bar{\varrho}(E)$

$$\bar{\mathcal{N}}_+^{(2)}(E) = \underbrace{\frac{L^2}{8\pi}\theta(E) + \frac{\sqrt{2}L}{4\pi}\sqrt{E}\theta(E)}_{\mathrm{non-int.}} - 2\frac{\sqrt{2}L}{4\pi}\sqrt{E}\theta(E) + 2\frac{\sqrt{2}L}{4\pi}(\sqrt{E+\alpha} - \sqrt{\alpha})\theta(E)$$



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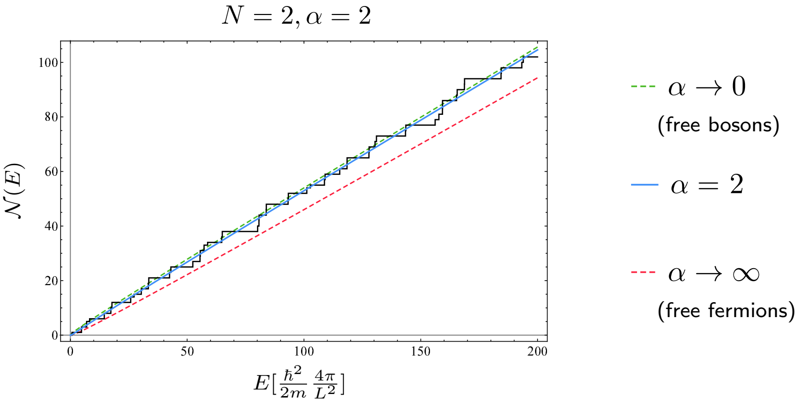

2 bosons, **counting function** $\mathcal{N}(E) = \int \mathrm{d}E\, \bar{\varrho}(E)$

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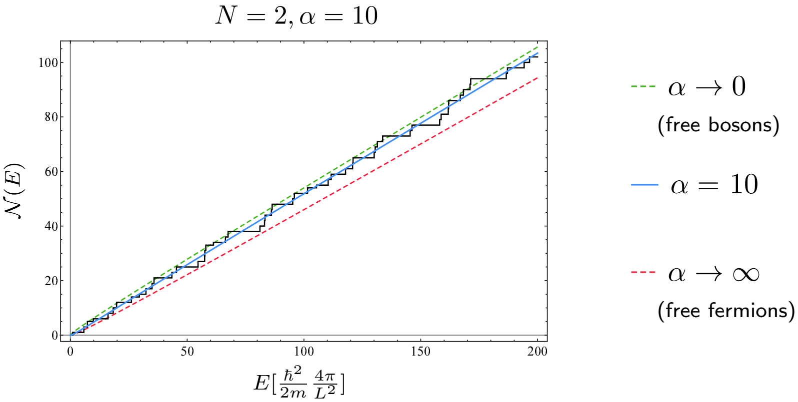

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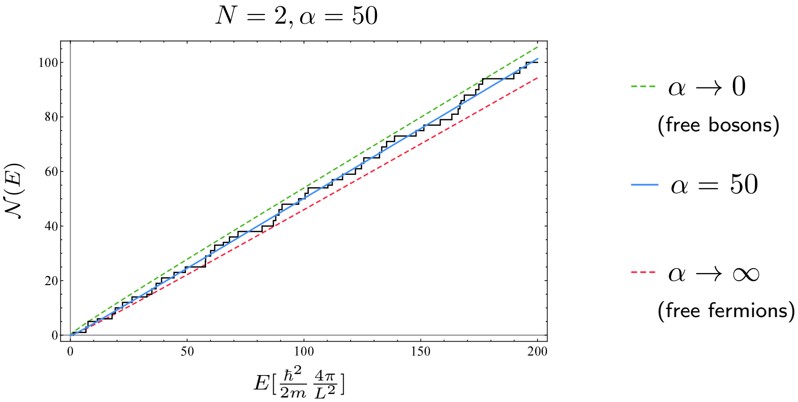

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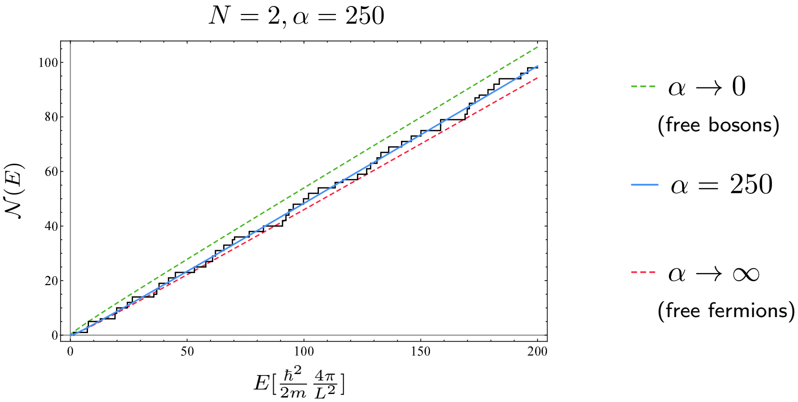

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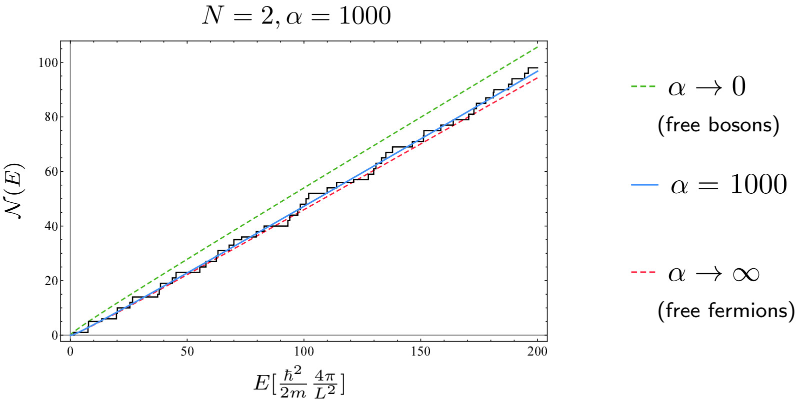

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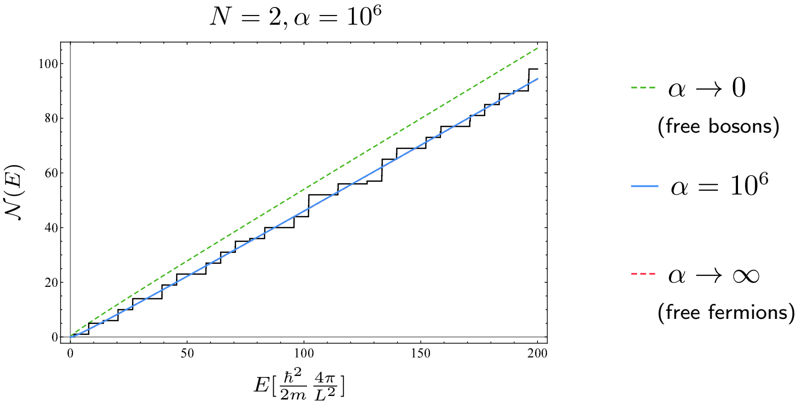

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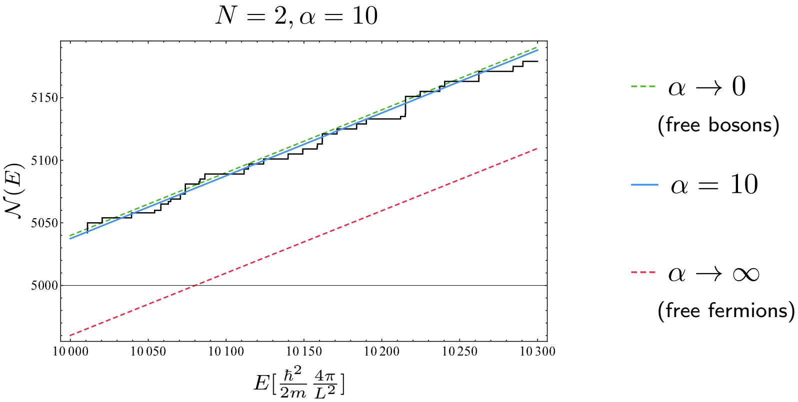

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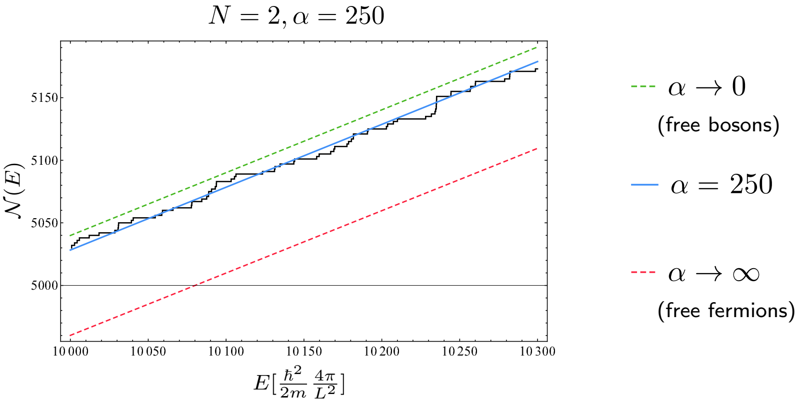

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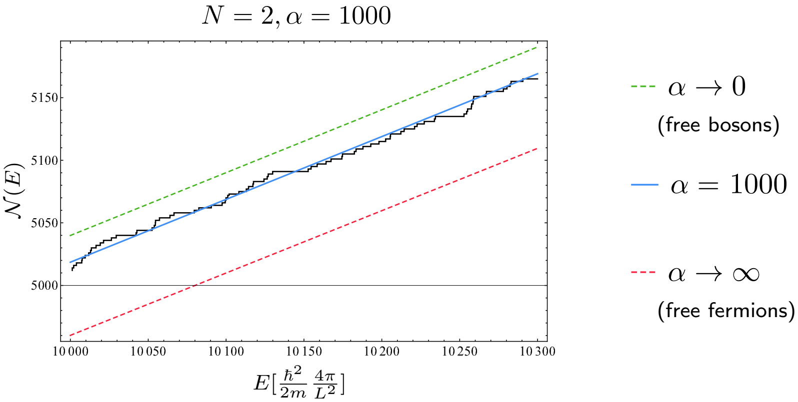


Overview

N bosons, "quantum cluster expansion"

path integral representation:

$$K^{(N)}(\mathbf{q}^{\mathrm{f}},\mathbf{q}^{\mathrm{i}};t)=\int_{\mathbf{q}^{\mathrm{i}}}^{\mathbf{q}^{\mathrm{f}}}\mathcal{D}[\mathbf{q}(s)]\prod_{k=1}^{N}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_0^t\frac{m}{2}[\dot{\mathbf{q}}_k(s)]^2\mathrm{d}s}\prod_{k<l}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int_0^t V(\mathbf{q}_k(s)-\mathbf{q}_l(s))\mathrm{d}s}$$

"Mayer functionals" f_{kl}:

$$1+f_{kl}:=\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int_0^t V(\mathbf{q}_k(s)-\mathbf{q}_l(s))\mathrm{d}s}$$

expand:

$$\prod_{k<l}(1+f_{kl})=1+\sum_{k<l}f_{kl}+\dots$$

$$\Downarrow$$

$$K^{(N)}(\mathbf{q}^{\mathrm{f}},\mathbf{q}^{\mathrm{i}};t)=K_0^{(N)}(\mathbf{q}^{\mathrm{f}},\mathbf{q}^{\mathrm{i}};t)+\sum_{k<l}K_0^{(N-2)}(\mathbf{q}^{\mathrm{f}}_{\overline{kl}},\mathbf{q}^{\mathrm{i}}_{\overline{kl}};t)\Delta K^{(2)}(\mathbf{q}^{\mathrm{f}}_{kl},\mathbf{q}^{\mathrm{i}}_{kl};t)+\dots$$

N bosons, "quantum cluster expansion"

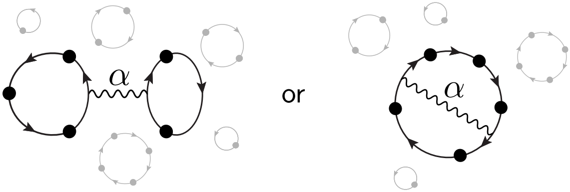
with permutations: $\int \mathrm{d}^N q K^{(N)}(P\mathbf{q}, \mathbf{q}; t)$

- $P = \mathrm{id}$:
  - · usual cluster expansion
  - · direct $\mathbf{p}$-integration in classical $Z(\beta)$

N bosons, "quantum cluster expansion"

with permutations: $\int \mathrm{d}^N q K^{(N)}(P\mathbf{q}, \mathbf{q}; t)$

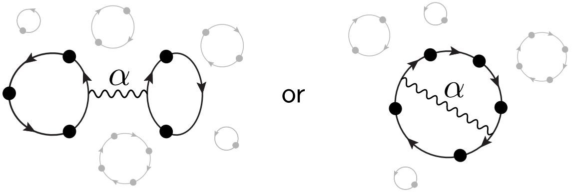
- $P \neq \mathrm{id}$:
  · need "quantum cluster expansion"
  · yields traces like



N bosons, "quantum cluster expansion"

with permutations: $\int \mathrm{d}^N q K^{(N)}(P\mathbf{q}, \mathbf{q}; t)$

- $P \neq \mathrm{id}$:
  - · need "quantum cluster expansion"
  - · yields traces like



  - · δ-int.: solution in t/β-domain involves special function (Owen's T-f.)
  - · but: $\mathcal{L}_\beta^{-1}[\ldots]$ has elementary solution!

N bosons, "quantum cluster expansion"

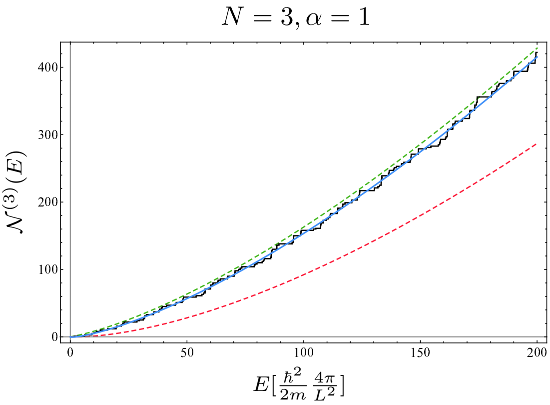
e.g. 3 bosons ($\frac{\hbar^2}{2mL^2} = \frac{1}{4\pi}$)

$$\bar{\mathcal{N}}_{0,+}^{(3)}(E) = \frac{2}{9\sqrt{\pi}}E^{3/2}\theta(E) + \frac{1}{\sqrt{8}}E\theta(E) + \frac{2}{\sqrt{27\pi}}E^{1/2}\theta(E)$$

$$\begin{aligned}\Delta_1\bar{\mathcal{N}}_{+}^{(3)}(E) = &-\frac{1}{2}E\theta(E) - \frac{\sqrt{2\alpha}}{\pi}E^{1/2}\theta E - \frac{8}{\sqrt{27\pi}}E^{1/2}\theta(E) \\ &- \frac{2\sqrt{\alpha}}{\sqrt{\pi}}\theta(E) + \frac{\sqrt{2}}{\pi}(E+\alpha)\arctan\left(\sqrt{\frac{E}{\alpha}}\right)\theta(E) \\ &+ \frac{8}{\pi^{3/2}}(E+\alpha)(3E+4\alpha)^{-1/2}\arctan\left(\sqrt{3+\frac{4\alpha}{E}}\right)\theta(E)\end{aligned}$$

N bosons, "quantum cluster expansion"

3 bosons, counting function $\mathcal{N}(E) = \int \mathbf{d}E\, \bar{\varrho}(E)$



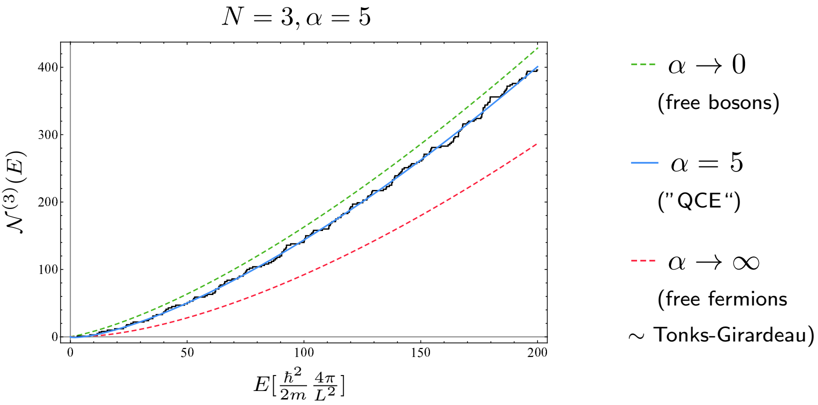
$\alpha \to 0$ (free bosons)

$\alpha = 1$ ("QCE")

$\alpha \to \infty$ (free fermions $\sim$ Tonks-Girardeau)

N bosons, "quantum cluster expansion"

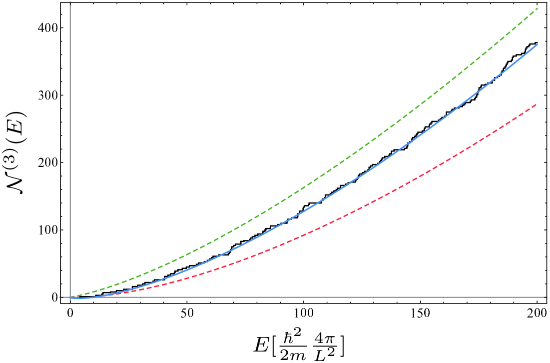
3 bosons, counting function $\mathcal{N}(E) = \int \mathbf{d}E\, \bar{\varrho}(E)$



N bosons, "quantum cluster expansion"

3 bosons, counting function $\mathcal{N}(E) = \int \mathbf{d}E\, \bar{\varrho}(E)$

$N = 3, \alpha = 25$



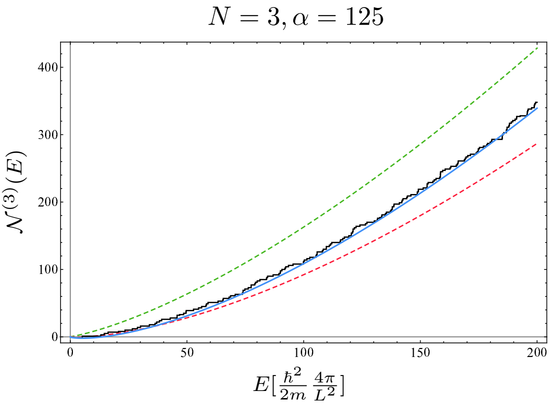
$\alpha \to 0$ (free bosons)

$\alpha = 25$ ("QCE")

$\alpha \to \infty$ (free fermions ∼ Tonks-Girardeau)

N bosons, "quantum cluster expansion"

3 bosons, counting function $\mathcal{N}(E) = \int \mathbf{d}E\, \bar{\varrho}(E)$



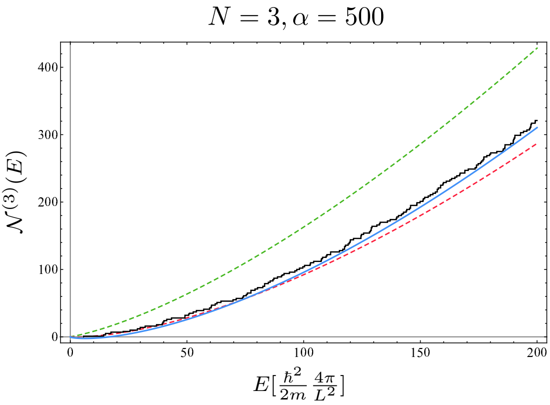
- - - $\alpha \to 0$
(free bosons)

— $\alpha = 125$
("QCE")

- - - $\alpha \to \infty$
(free fermions
$\sim$ Tonks-Girardeau)

N bosons, "quantum cluster expansion"

3 bosons, counting function $\mathcal{N}(E) = \int \mathbf{d}E\, \bar{\varrho}(E)$



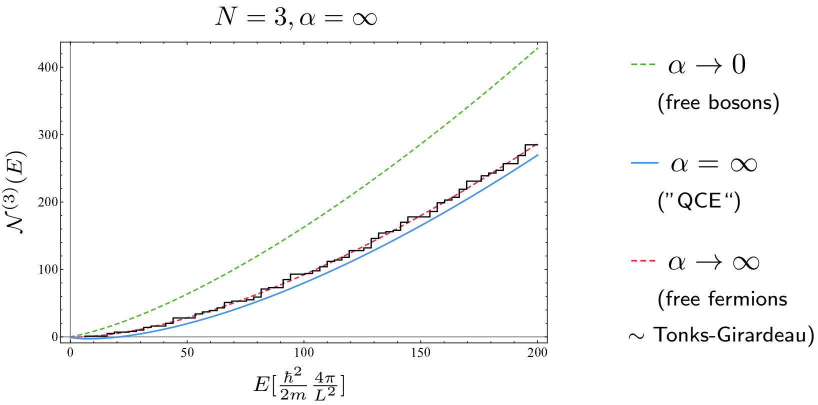
$\alpha \to 0$ (free bosons)

$\alpha = 500$ ("QCE")

$\alpha \to \infty$ (free fermions $\sim$ Tonks-Girardeau)

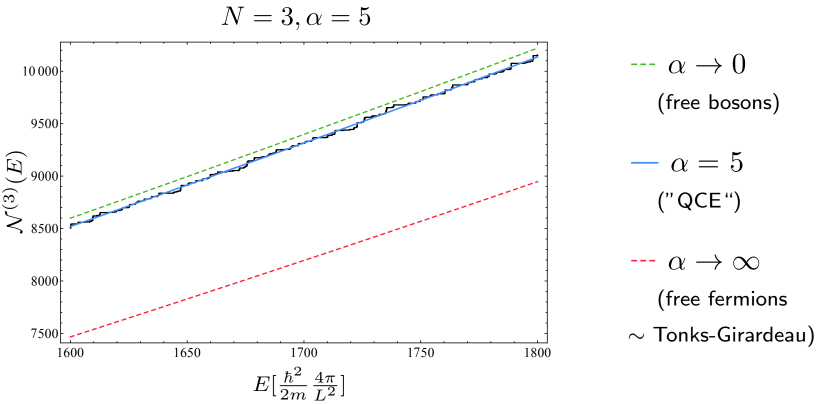
N bosons, "quantum cluster expansion"

3 bosons, counting function $\mathcal{N}(E) = \int \mathbf{d}E \, \bar{\varrho}(E)$



N bosons, "quantum cluster expansion"

3 bosons, counting function $\mathcal{N}(E) = \int \mathbf{d}E\, \bar{\varrho}(E)$



N bosons, "quantum cluster expansion"

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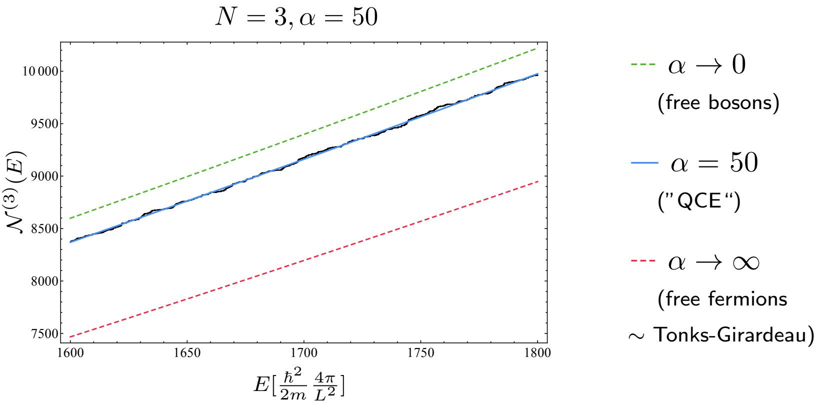

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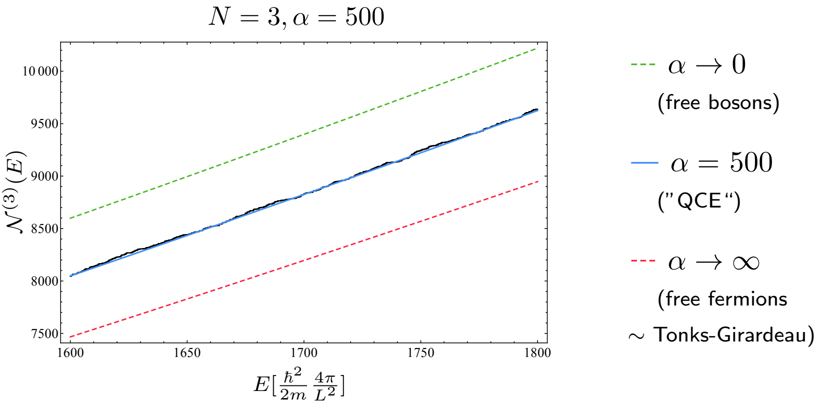
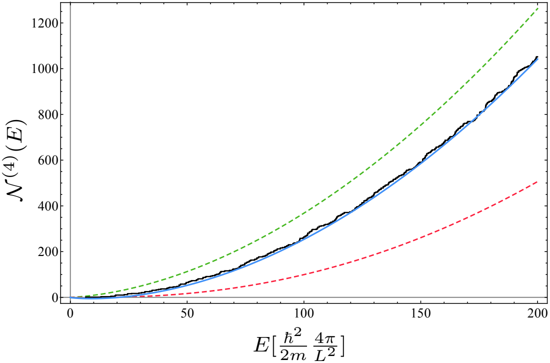


N bosons, "quantum cluster expansion"

more bosons (keep αN^2 constant)

$N = 4, \alpha = 6.25$



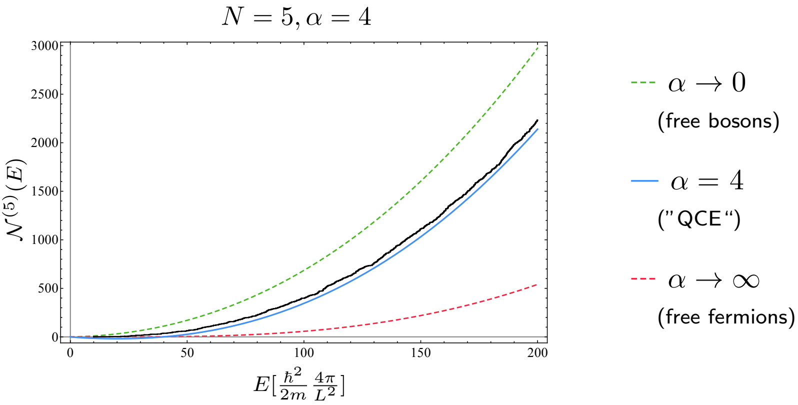
$\alpha \to 0$ (free bosons)

$\alpha = 6.25$ ("QCE")

$\alpha \to \infty$ (free fermions)

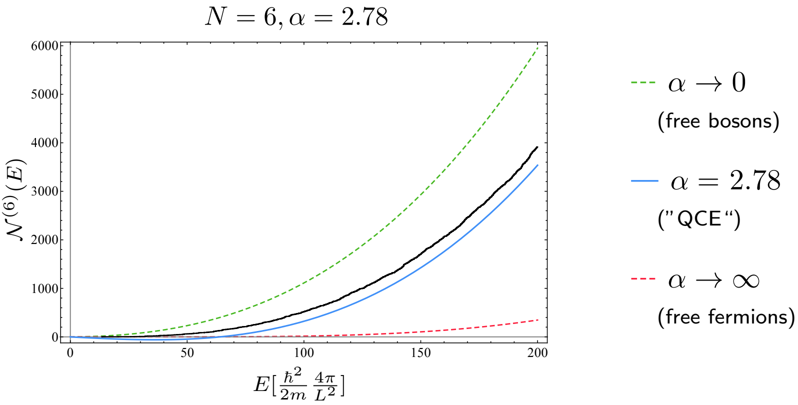
N bosons, "quantum cluster expansion"

more bosons (keep αN^2 constant)



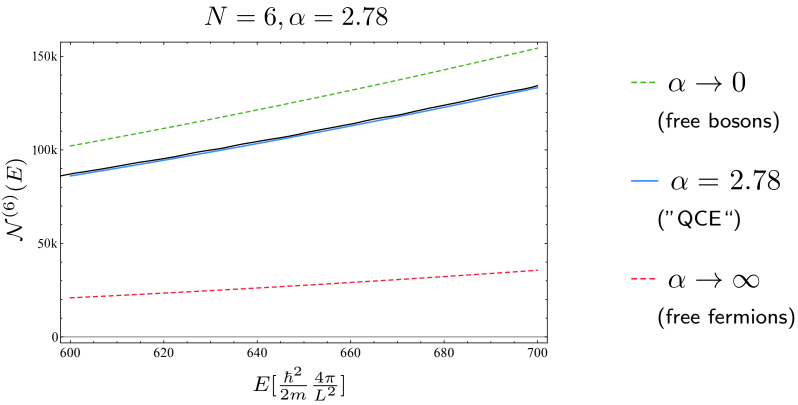
N bosons, "quantum cluster expansion"

more bosons (keep αN^2 constant)



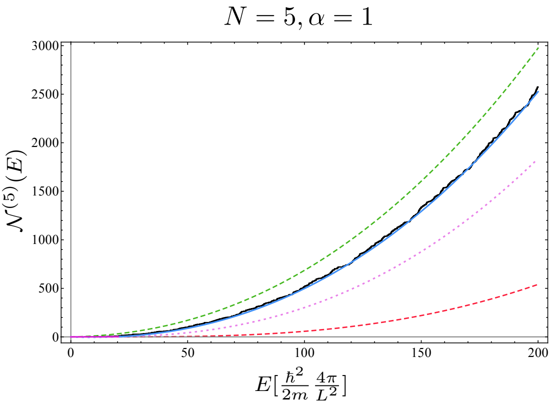
N bosons, "quantum cluster expansion"

more bosons (keep αN^2 constant)



N bosons, "quantum cluster expansion"

5 bosons, counting function $\mathcal{N}(E) = \int \mathbf{d}E\, \bar{\varrho}(E)$



- - - $\alpha \to 0$ (free bosons)

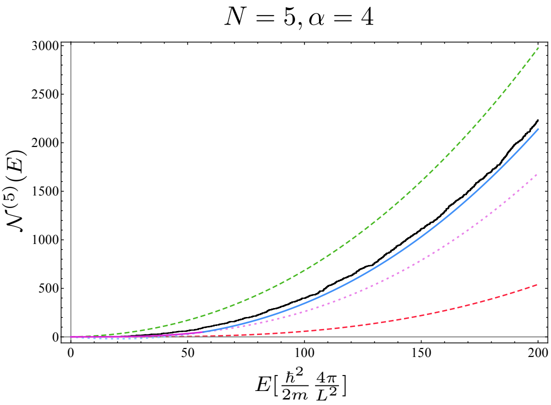
— $\alpha = 1$ ("QCE")

— $\alpha = 1$ ("QCE" of TG)

- - - $\alpha \to \infty$ (free fermions)

N bosons, "quantum cluster expansion"

5 bosons, counting function $\mathcal{N}(E) = \int \mathbf{d}E \, \bar{\varrho}(E)$



$\alpha \to 0$ (free bosons)

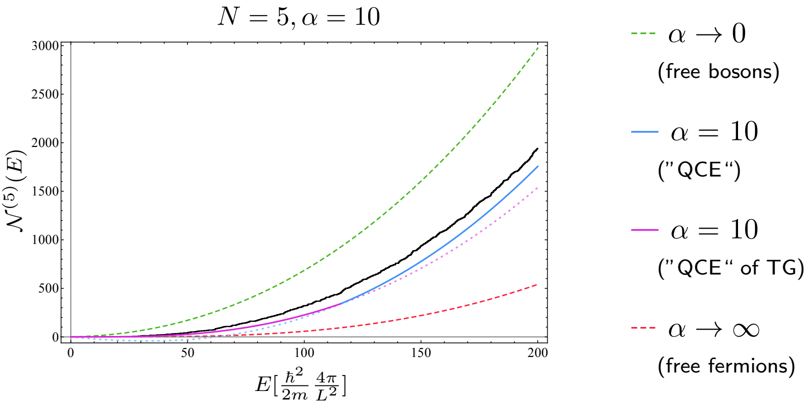
$\alpha = 4$ ("QCE")

$\alpha = 4$ ("QCE" of TG)

$\alpha \to \infty$ (free fermions)

N bosons, "quantum cluster expansion"

5 bosons, counting function $\mathcal{N}(E) = \int \mathbf{d}E\, \bar{\varrho}(E)$



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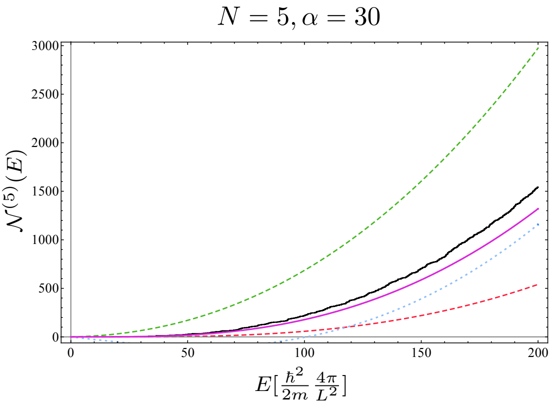


$\alpha \to 0$ (free bosons)

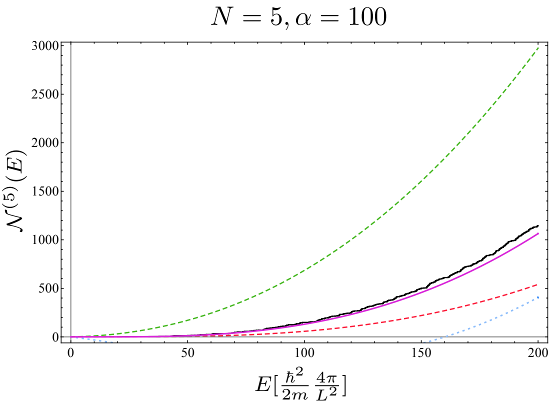
$\alpha = 30$ ("QCE")

$\alpha = 30$ ("QCE" of TG)

$\alpha \to \infty$ (free fermions)

N bosons, "quantum cluster expansion"

5 bosons, counting function $\mathcal{N}(E) = \int \mathbf{d}E\, \bar{\varrho}(E)$



$\alpha \to 0$
(free bosons)

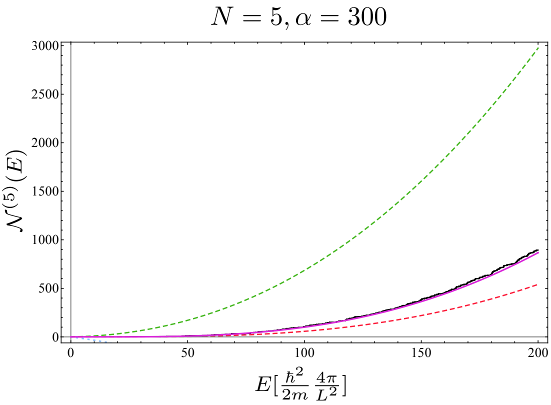
$\alpha = 100$
("QCE")

$\alpha = 100$
("QCE" of TG)

$\alpha \to \infty$
(free fermions)

N bosons, "quantum cluster expansion"

5 bosons, counting function $\mathcal{N}(E) = \int \mathbf{d}E\, \bar{\varrho}(E)$



$\alpha \to 0$ (free bosons)

$\alpha = 300$ ("QCE")

$\alpha = 300$ ("QCE" of TG)

$\alpha \to \infty$ (free fermions)

Thermodynamics of QCE

Scaling of canonical partition function in QCE (general)

$$Z(V,\beta,N,\alpha) = \sum_{l=1}^{N} [\underbrace{z_l}_{\mathrm{non-int.}} + \underbrace{\Delta z_l(\beta\alpha)}_{\mathrm{QCE}}] \left(\frac{V}{\lambda_T^D}\right)^l$$

Thermodynamics of QCE

Scaling of canonical partition function in QCE (general)

$$Z(V,\beta,N,\alpha) = \sum_{l=1}^{N} [\underbrace{z_l}_{\text{non}-\text{int.}} + \underbrace{\Delta z_l(\beta\alpha)}_{\text{QCE}}] \left(\frac{V}{\lambda_T^D}\right)^l$$

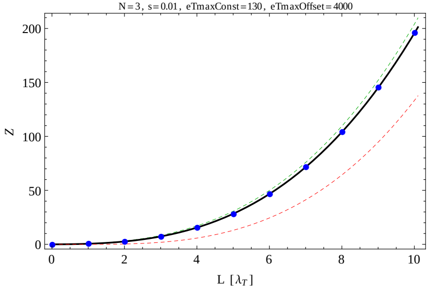
$$\Rightarrow P(V,\beta,N,\alpha) = \frac{k_\text{B}T}{V} \frac{\sum_{l=1}^{N} l[z_l + \Delta z_l(\beta\alpha)]\left(\frac{V}{\lambda_T^D}\right)^l}{\sum_{l=1}^{N} [z_l + \Delta z_l(\beta\alpha)]\left(\frac{V}{\lambda_T^D}\right)^l}$$

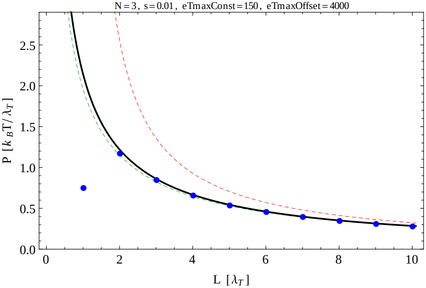
$\rightarrow$ simple polynomial / rational structure remained

Thermodynamics of QCE

EOS for Lieb-Liniger model $N = 3$

$\beta\alpha =$ 0.01 0.05 0.1 0.25 0.5





QCE, Tonks-Girardeau, id. Bose Gas, exact

Thermodynamics of QCE

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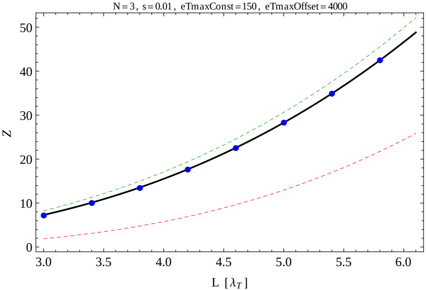

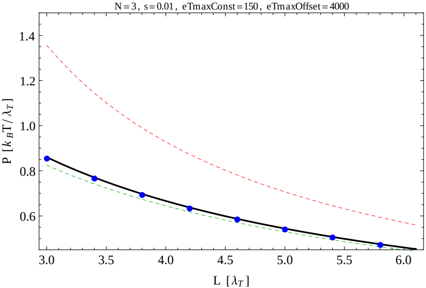

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Thermodynamics of QCE

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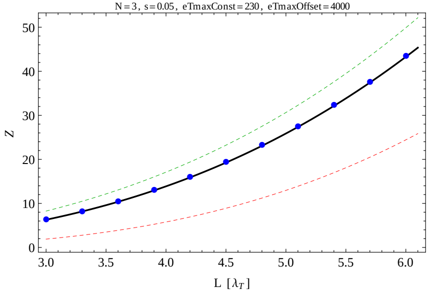
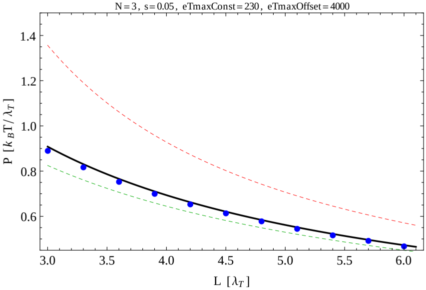


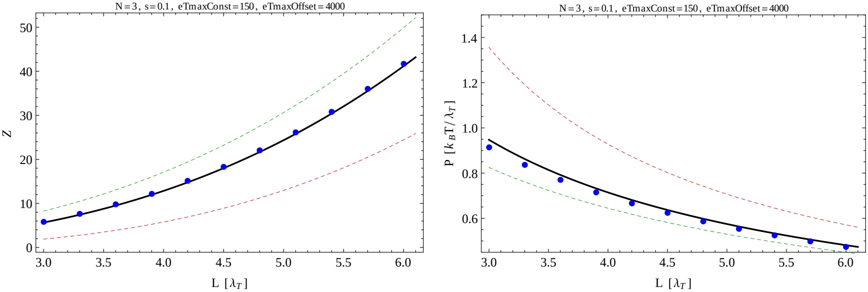


QCE, Tonks-Girardeau, id. Bose Gas, exact

Thermodynamics of QCE

EOS for Lieb-Liniger model $N = 3$

$\beta\alpha =$ 0.01 0.05 **0.1** 0.25 0.5

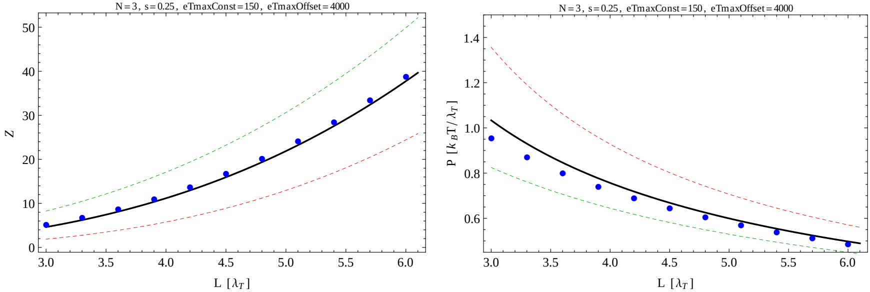


QCE, Tonks-Girardeau, id. Bose Gas, exact

Thermodynamics of QCE

EOS for Lieb-Liniger model $N = 3$

$\beta\alpha =$ 0.01 0.05 0.1 **0.25** 0.5

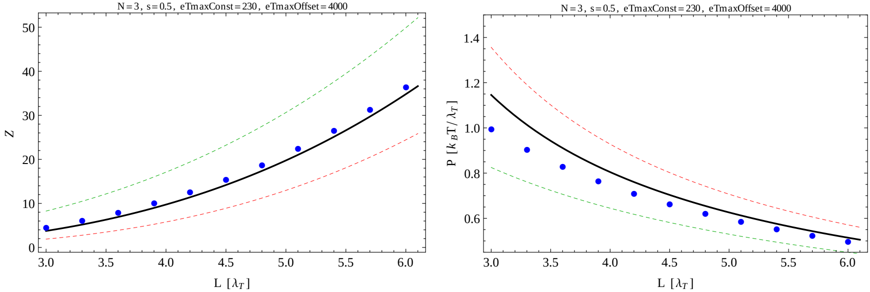


QCE, Tonks-Girardeau, id. Bose Gas, exact

Thermodynamics of QCE

EOS for Lieb-Liniger model $N = 3$

$\beta\alpha =$ 0.01 0.05 0.1 0.25 **0.5**



QCE, Tonks-Girardeau, id. Bose Gas, exact

Thermodynamics of QCE

EOS for Lieb-Liniger model $N = 3$

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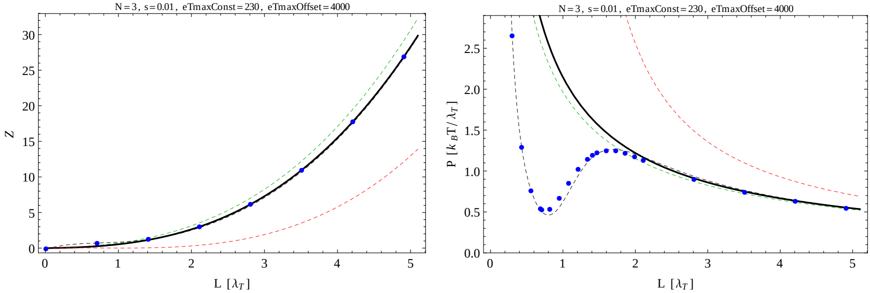

QCE, Tonks-Girardeau, id. Bose Gas, exact

Dashed: $Z = \mathrm{e}^{-\beta E_0} + \mathrm{e}^{-\beta E_1} \sum_l \bar{z}_l (V/\lambda_T)^l$ $\rightarrow$ fully analytic!

Thermodynamics of QCE

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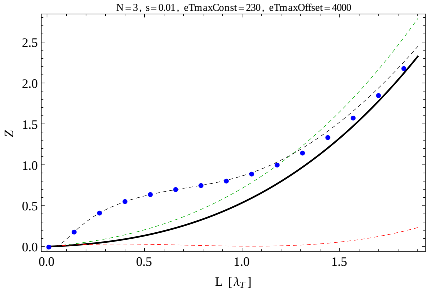

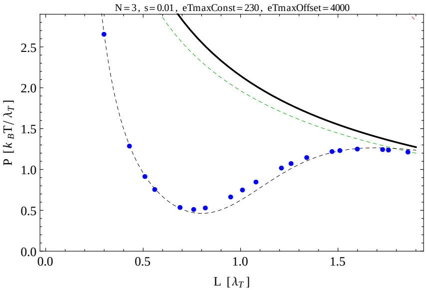

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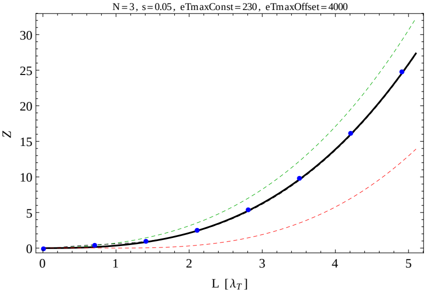

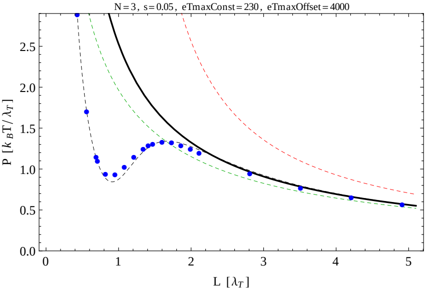

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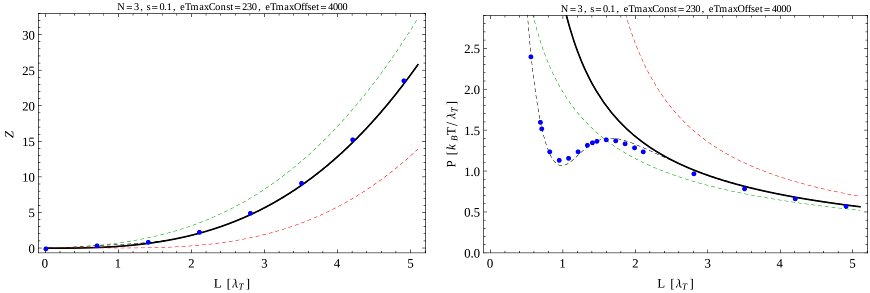

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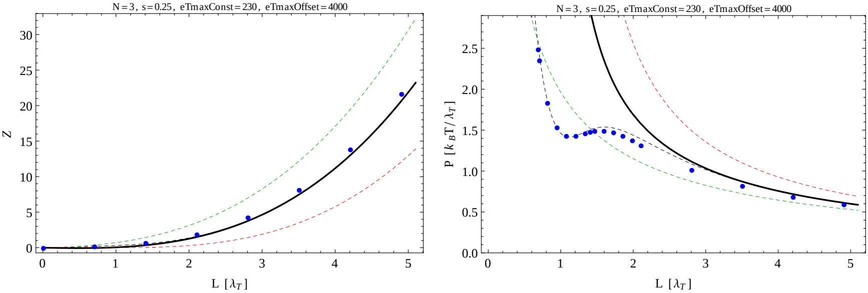

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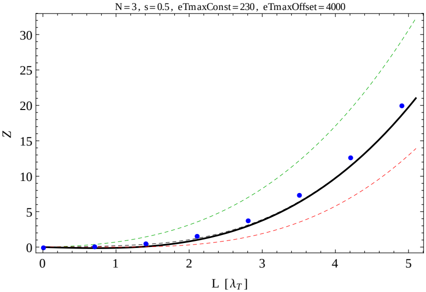

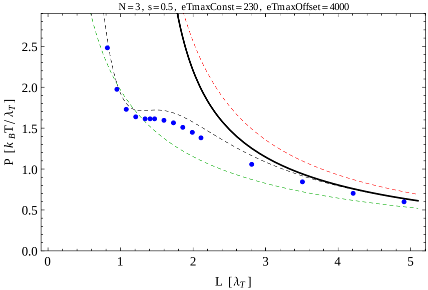

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Conclusion and Outlook

Constructed analytic Weyl law / finite EOS for ...

- 2 bosons with contact interaction (exact)
  $\rightarrow$ agreement with numerics
- N bosons with contact interaction in terms of a first order "quantum" cluster expansion
  $\rightarrow$ agreement in regimes of "weak" and infinite interaction **or** high excitation, "high" temperature
  $\rightarrow$ improvement by splitting off lowest level

Projects:

- use 2-body info to all orders
- apply techniques to dynamical impurity model
  (N non-int. fermions + 1 int. particle $\rightarrow$ is QCE "complete"?)
- N spin-1/2 fermions (Luttiger liquid physics)
- Harmonic confinement (not solvable!)
- 3,5,7,... body terms from Yang-Yang solution of Lieb-Linniger.

Thermodynamics of QCE

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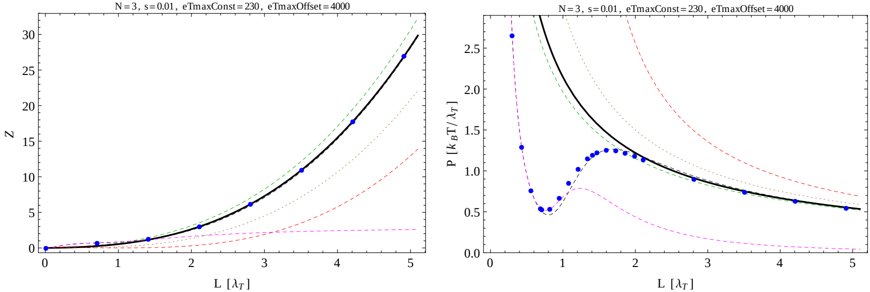

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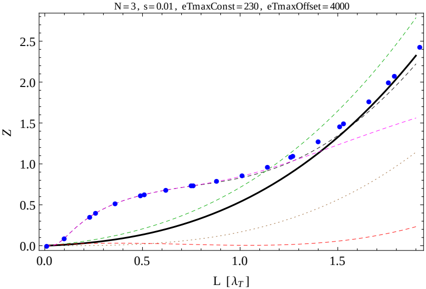

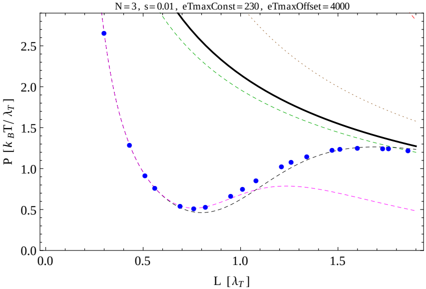

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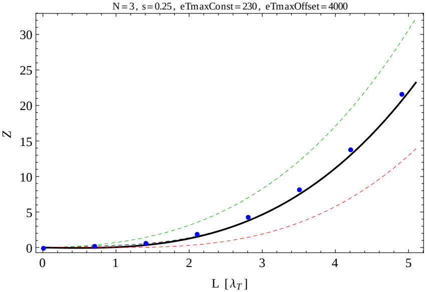

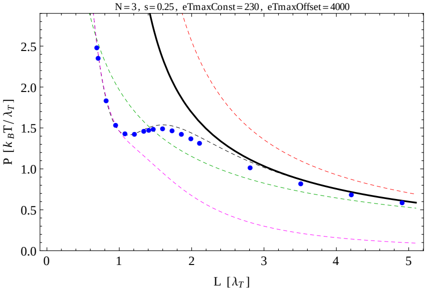

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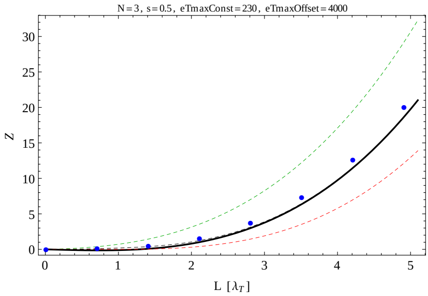

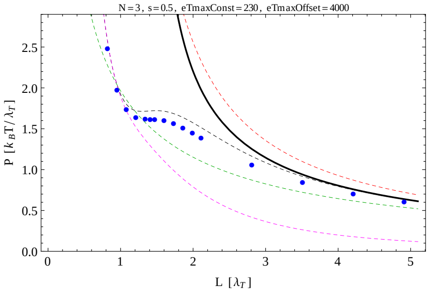

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