Link Analysis

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Rankings may be composed (e.g., by linear combination): this is called *rank aggregation*.

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- *Selection*: it selects, from the set D of all available documents, a subset $S(q)$ of documents that satisfy q;
- *Ranking*: it establishes a total order on $S(q)$ determining how the results should be presented to the user.

The Web as a Graph

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This is called the *Web graph*.

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Depending on whether the scoring (ranking) function depends or not on the query, and whether it depends or not on the text of the page (or only on its links):

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	Query-dependent (dynamic)	**Query-independent (static)**
Text-based	IR	-
Link-based	e.g., HITS	e.g., PageRank

Link Analysis — Problem and assumptions

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- **Basic assumption:** A link is a way to confer importance.

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- it is static, so it can be computed beforehand (not at query time)
- it can be computed efficiently
- it is (used to be) the main ranking technique used at Google.

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Problem with this solution: Formation of oligopolies that "suck away" all money from the system, without ever giving it back.

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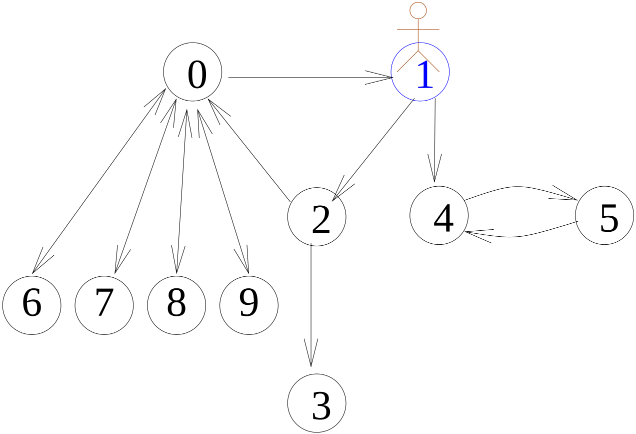
Another problem: What should the dangling nodes do? (A *dangling node* is one that has no out-neighbors)

Dangling nodes pay, as every other node, $1 - \alpha$ in taxes, and distribute α to the nodes according to a fixed *dangling-node distribution* u.

PageRank: the Web-Surfer Metaphor



A surfer is wandering about the web. . .

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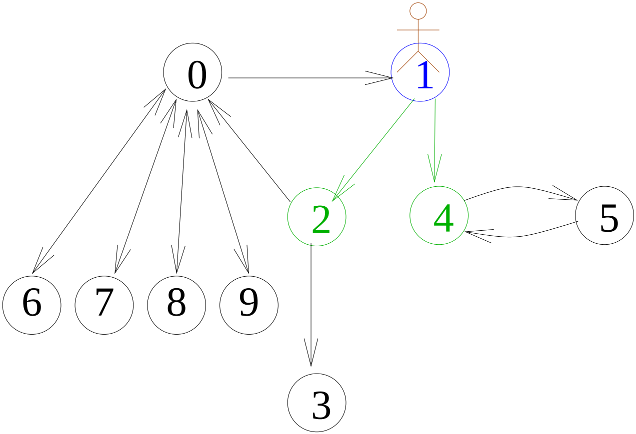

At each step, with probability α (s)he chooses the next page by clicking on a random link...

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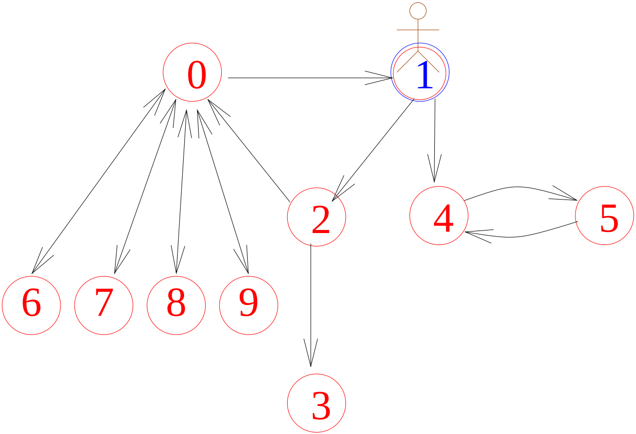

... with probability $1 - \alpha$, (s)he jumps to a random node (chosen uniformly or according to a fixed distribution, the *preference vector*)

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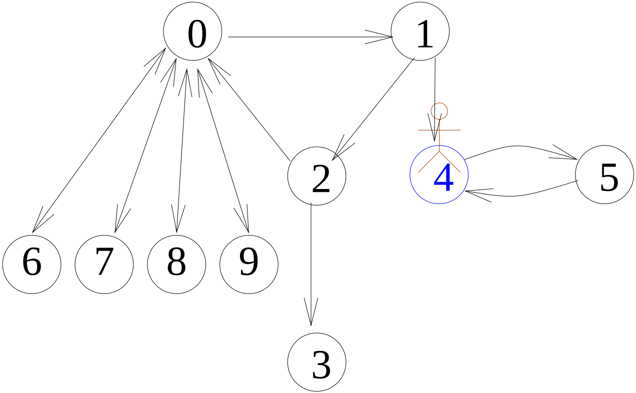

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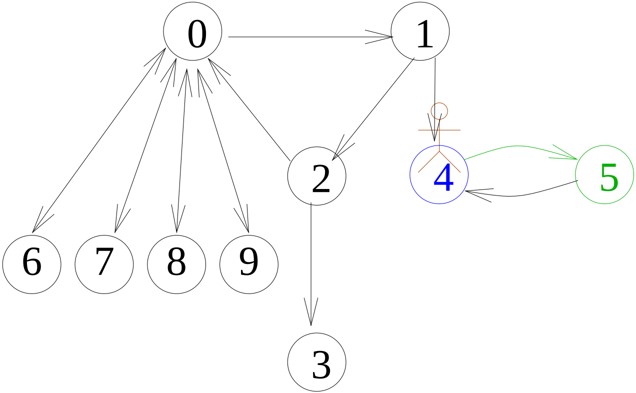

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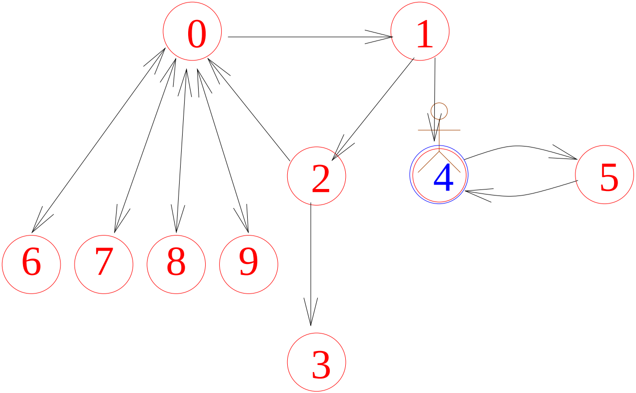

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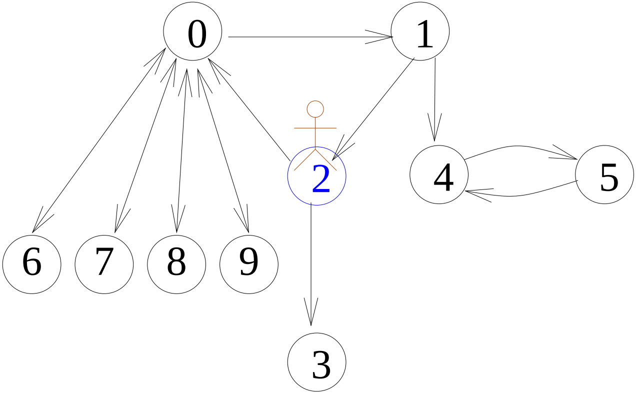

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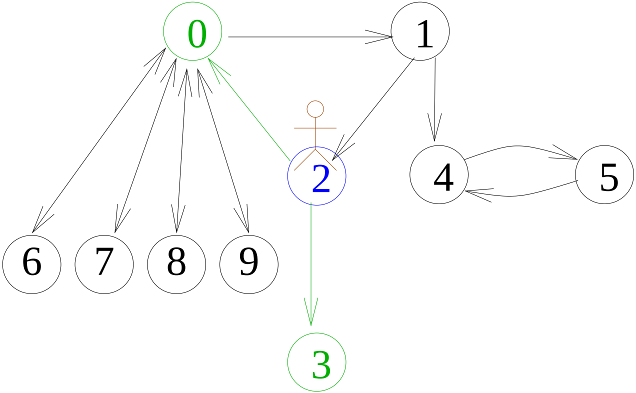

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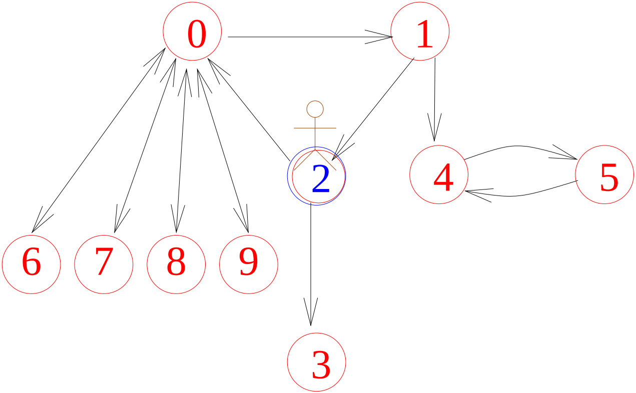

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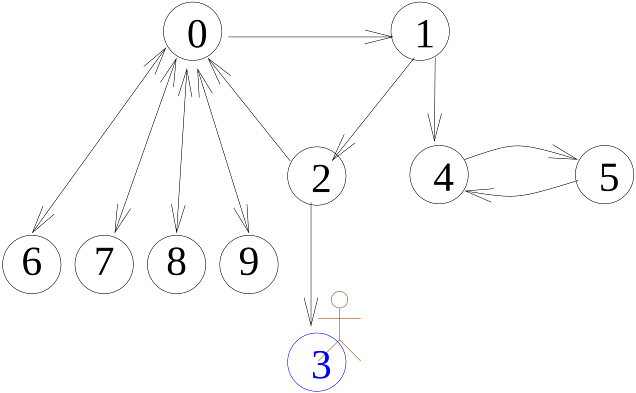

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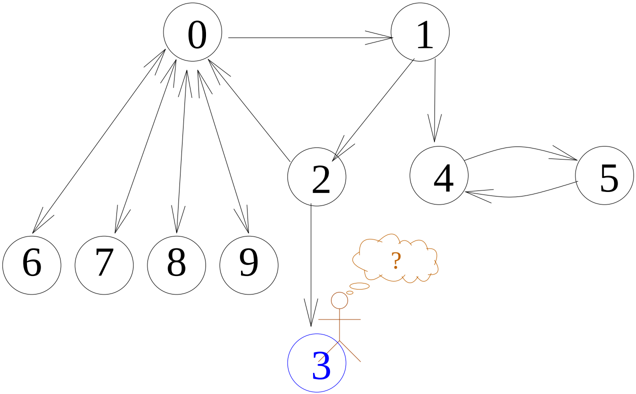

What if (s)he reaches a node with no outlinks (a *dangling node*)?

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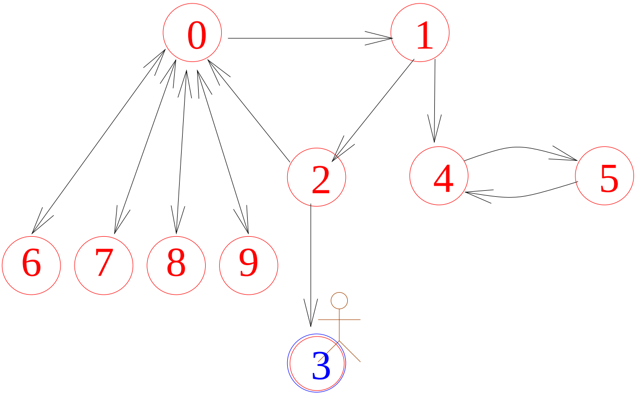

In that case, (s)he jumps to a random node *with probability 1*.

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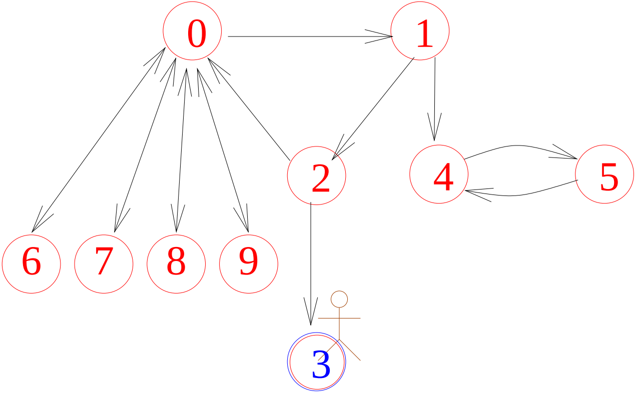

The PageRank of a page is the average fraction of time spent by the surfer on that page.

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- the *web graph* G;
- the *preference vector* v;
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- the *damping factor* α.

How does PageRank depends on each of these factors? What happens at limit values (e.g., $\alpha \to 1$)?

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where $f(-)$ is a suitable *damping function* that goes to zero sufficiently fast [Baeza–Yates, Boldi & Castillo 2006].

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Theorem

The n-th approximation of PageRank computed by the Power Method with damping factor α and starting vector v coincides with the n-th degree Maclaurin polynomial of PageRank evaluated in α.

$$vM^n = v + v\sum_{k=1}^{n}\alpha^k\left(P^k - P^{k-1}\right).$$

One α to rule them all...

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Corollary

The difference between the k-th and the $(k-1)$-th approximation of PageRank (as computed by the Power Method with starting vector v), divided by α^k, is the k-th coefficient of the power series of PageRank.

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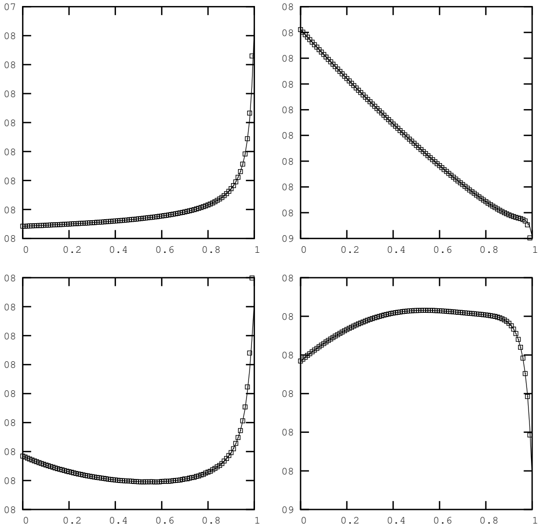
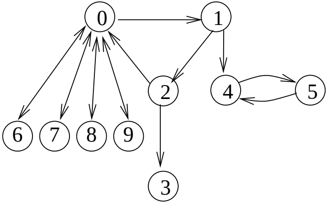
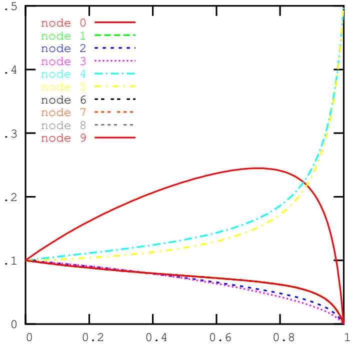
Even more is true, of course: using standard series derivation techniques, one can approximate the k-th derivative.

Some typical behaviours



An example





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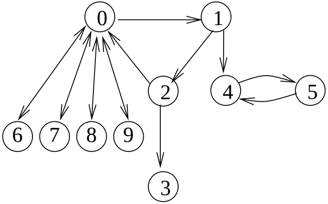

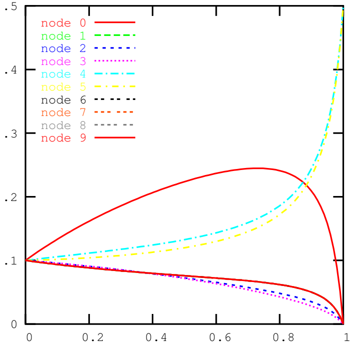

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The magic value $\alpha = 0.85$

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- ...numeric instability arises when α is too close to 1...
- ...yet, we believe that understanding how $r(\alpha)$ changes when α is modified is important.

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Since r is a coordinatewise bounded function defined on $[0, 1)$, the limit

$$r^* = \lim_{\alpha \to 1^-} r$$

exists.

A ready-made solution

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In fact, since the *resolvent* $(I/\alpha - P)$ has a Laurent expansion around 1 in the largest disc not containing $1/\lambda$ for another eigenvalue λ of P, PageRank is analytic in the same disc; a standard computation yields

$$(1-\alpha)(1-\alpha P)^{-1} = P^* - \sum_{n=0}^{\infty} \left(\frac{\alpha - 1}{\alpha}\right)^{n+1} Q^{n+1},$$

where $Q = (I - P + P^*)^{-1} - P^*$ and

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What makes r^* different from other limit distributions? How can we describe its structure?

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A characterisation theorem

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Corollary

Assume $u = \mathbf{1}/n$. Then:

1. *if G contains a bucket then a node is recurrent for P iff it is a bucket;*
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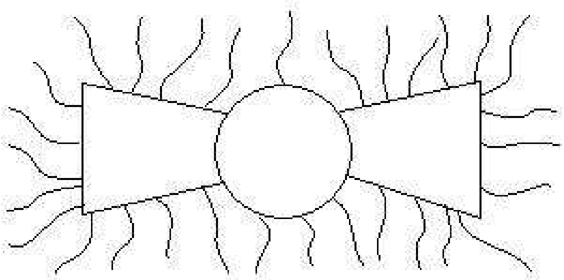
Theorem

1. *If a bucket of G is reachable from the support of u then a node is recurrent for P iff it is a bucket of G;*
2. *if no bucket of G is reachable from the support of u, all nodes reachable from the support of u form a bucket component of P; hence, a node is recurrent for P iff it is in a bucket component of G or it is reachable from the support of u.*

Bowtie

As a consequence, when $\alpha \to 1$, all PageRank concentrates in a bunch of pages that live in the rightmost part of the bowtie [Kumar et al., '00]:



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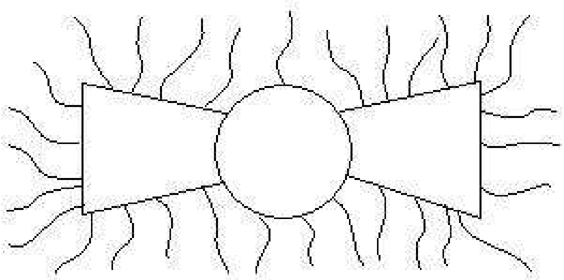

$r(\alpha)$ becomes meaningless as $\alpha \to 1$!

Interpretation

The statement of the previous theorem may seem a bit unfathomable. The essence, however, could be stated as follows: except for strongly connected graphs, or graphs whose terminal components are dangling, **the recurrent nodes are exactly the buckets** (unless we are in the very pathological case in which no bucket is reachable from the support of u).

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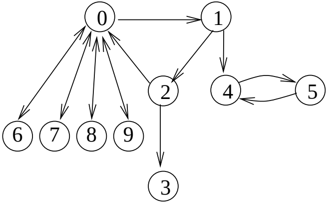
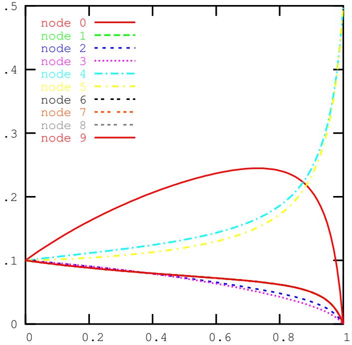
. . . and if you want the dire truth, there is an explicit formula in [Avrachenkov, Litvak & Kim 2006].

An example





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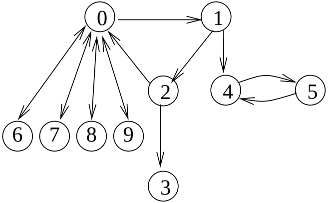

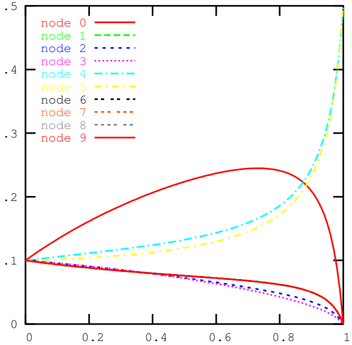

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Approximating them is also not difficult, since we have Maclaurin polynomials ($[\![r^{(k)}(\alpha)]\!]_t$ is the polynomial of order t):

Theorem

If $t \geq k/(1-\alpha)$,

$$\left\|r^{(k)}(\alpha) - [\![r^{(k)}(\alpha)]\!]_t\right\| \leq \frac{\delta_t}{1-\delta_t}\left\|[\![r^{(k)}(\alpha)]\!]_t - [\![r^{(k)}(\alpha)]\!]_{t-1}\right\|,$$

where

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Also TotalRank is a special case of the general ranking technique of [Baeza–Yates, Boldi & Castillo 2006]. The two damping functions for TotalRank and PageRank are:

$$\begin{aligned} d_T(\ell) &= \frac{1}{(t+1)(t+2)} \\ d_P(\ell) &= (1-\alpha)\alpha^\ell. \end{aligned}$$

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The average path length of the Web is about 20, and $\alpha^*(20) \approx .85\ldots$

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- ...but the two versions are *very different!*: On a 100 million pages snapshot of the `.uk` domain, Kendall's τ is $\approx .25$ for a topic-based v and $u = \mathbf{1}/n$! [Boldi *et al.* 2006]

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Using the Sherman–Morrison formula it is possible to make the dependence on v and u explicit, and sort out what happens in the strongly preferential case.

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The notion appears in [Del Corso, Gullì & Romani 2004] and it has been used in [McSherry 2005; Fogaras, Rácz, Csalogány & Sarlós 2005] (actually, as *the* definition of PageRank).

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Using this formula, once the pseudoranks for certain distributions have been computed, it is possible to compute PageRank using any *convex combination* of such distributions as preference and dangling-node distribution.

Another evident feature of the above formula is that the dependence on the dangling-node distribution is *not linear*, so *we cannot expect strongly preferential PageRank to be linear in* v.

The strongly preferential case

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Nonetheless, if we fix $u = v$ and simplify the resulting formula (getting back the formula obtained by Del Corso, Gullì and Romani)...

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Assuming that $v = \lambda x + (1-\lambda)y$, we have

$$r = r_{\lambda x + (1-\lambda)y}(\alpha) \quad \propto \quad \lambda\widetilde{x}(\alpha) + (1-\lambda)\widetilde{y}(\alpha)$$

Alternatives to PageRank

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- HITS (Kleinberg)
- SALSA (Lempel, Moran), a variant of HITS (not covered here)

HITS

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Differently from PageRank it is *not query independent*.

HITS (cont'd)

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- the authoritativeness/hubbiness scores are computed for the pages in G_q

HITS — Phase 1

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G_q is obtained as follows:

- the set S_q of the top k pages relative to q are obtained using some techniques (e.g., BM25)
- for each $x \in S_q$, all nodes in $N^+(x)$ are added
- for each $x \in S_q$, at most h nodes of $N^-(x)$ are added

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h_x(t+1) &\propto \sum_{x \to y} a_y(t) \\
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The $\propto$ is necessary to avoid divergence (the scores are normalized at every iteration).

HITS — In practice

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- its dynamic nature (requiring computation at query time)
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It was supposedly used by Teoma (later Ask.com).