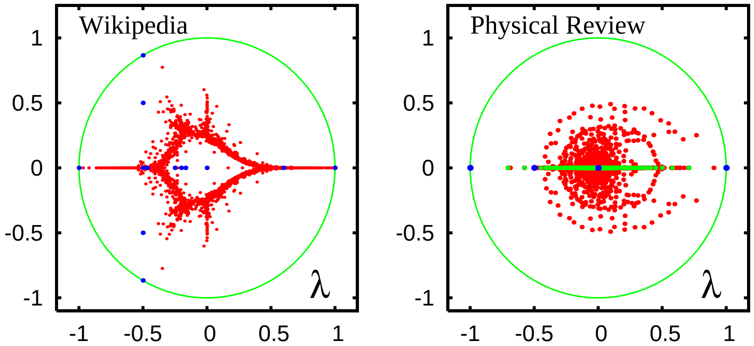

Spectral properties of Google matrix

Lecture 2

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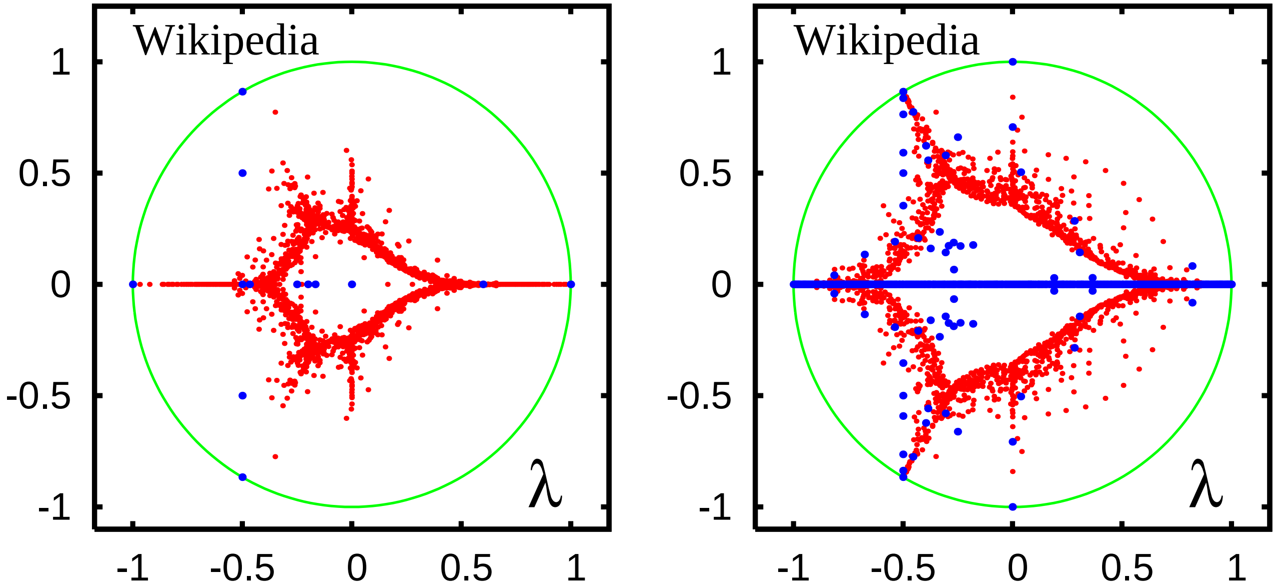
Network analysis and applications   Luchon, June 21 - July 5, 2014

Contents

Spectrum of Wikipedia 3
Eigenvectors of Wikipedia 4
Chirikov Standard map 9
Perron-Frobenius matrix for chaotic maps 10
Eigenvalues 12
Complex density of states 13
Eigenvectors 14
Diffuson modes 19
Extrapolation of eigenvalues 22
Strong chaos 23
Separatrix map 26
Poincaré recurrences 27
Phase space localization 29
References 30

Spectrum of Wikipedia

Wikipedia 2009 : $N = 3282257$ nodes, $N_\ell = 71012307$ network links.

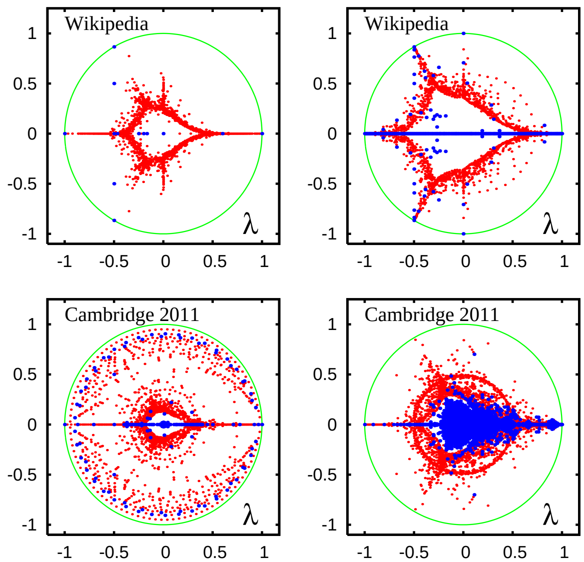


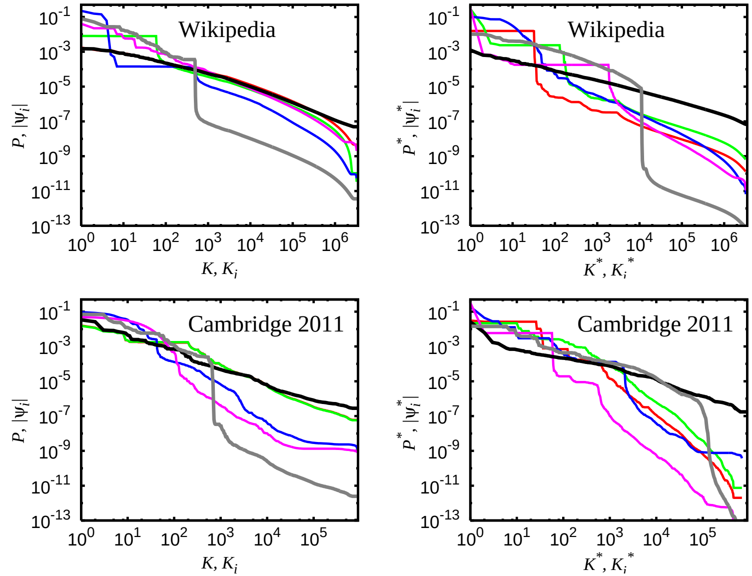
spectrum of S, $N_s = 515$

spectrum of S^*, $N_s = 21198$

$n_A = 6000$ for both cases

Eigenvectors of Wikipedia



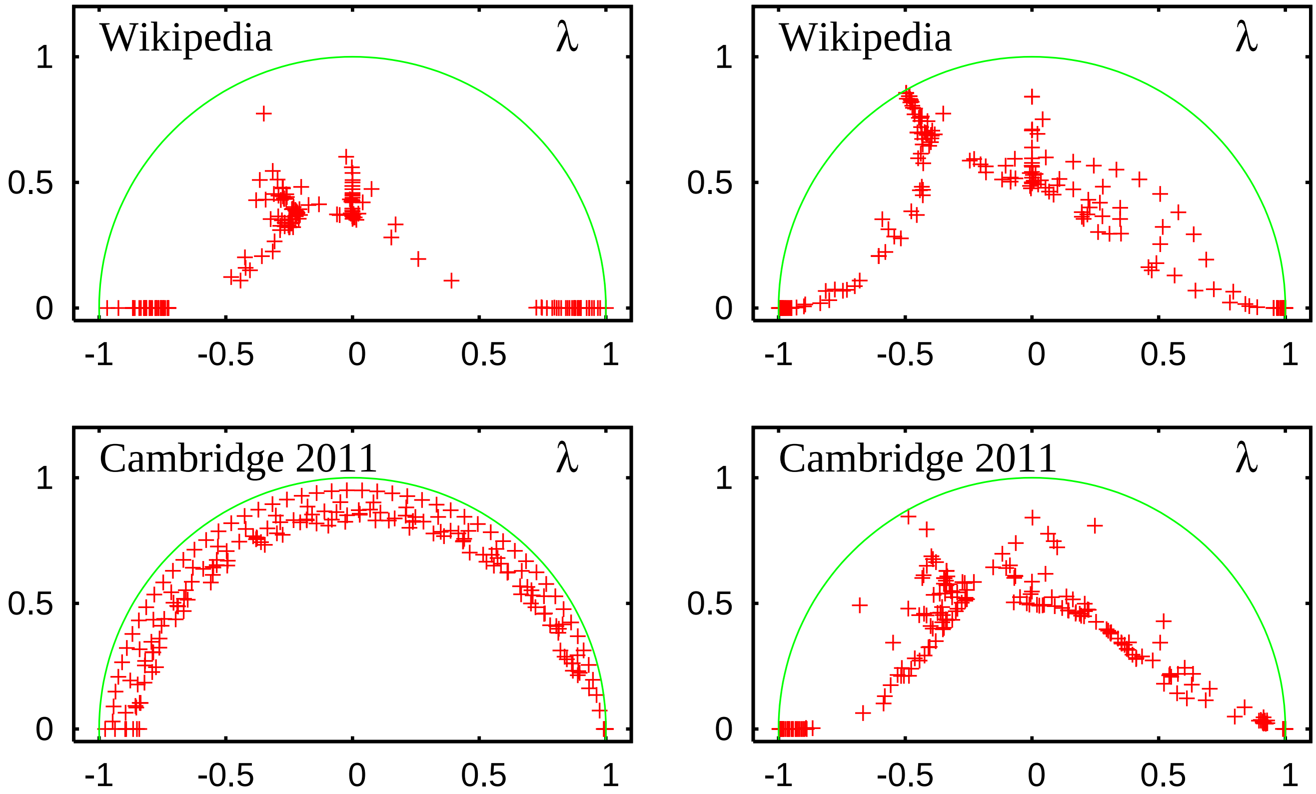


left (right): PageRank (CheiRank)

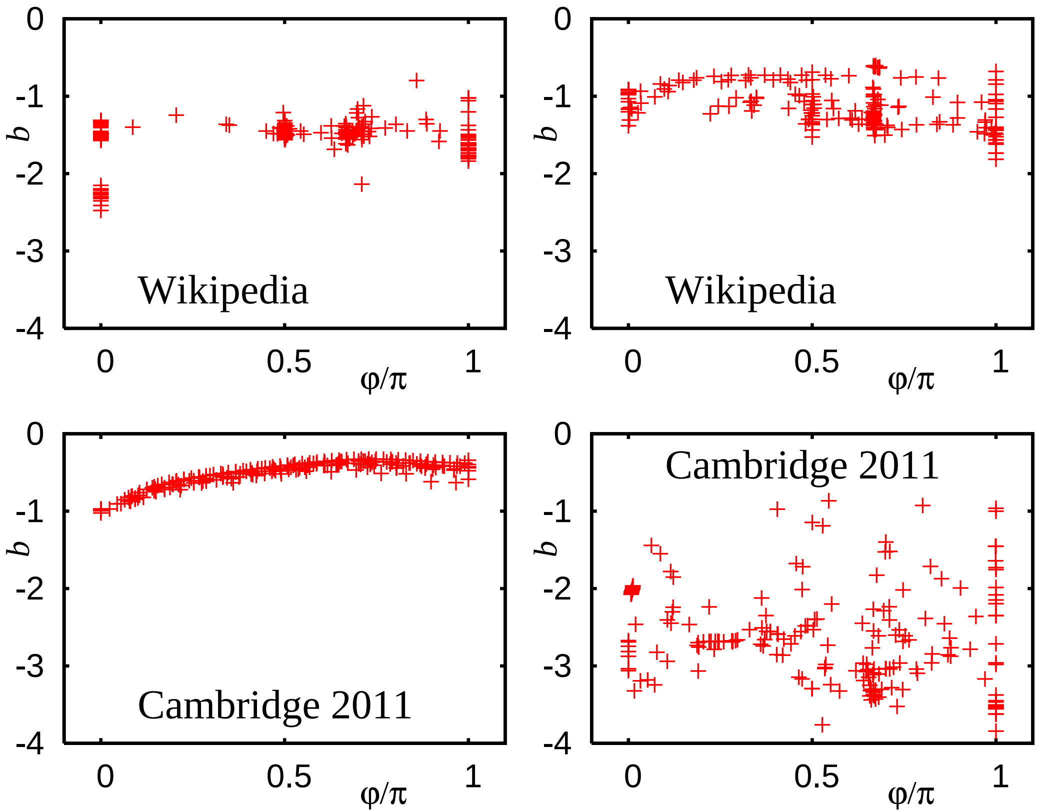
black: PageRank (CheiRank) at $\alpha = 0.85$
grey: PageRank (CheiRank) at $\alpha = 1 - 10^{-8}$
red and green: first two core space eigenvectors

blue and pink: two eigenvectors with large imaginary part in the eigenvalue

Detail study of 200 selected eigenvectors
with eigenvalues “close” to the unit circle:



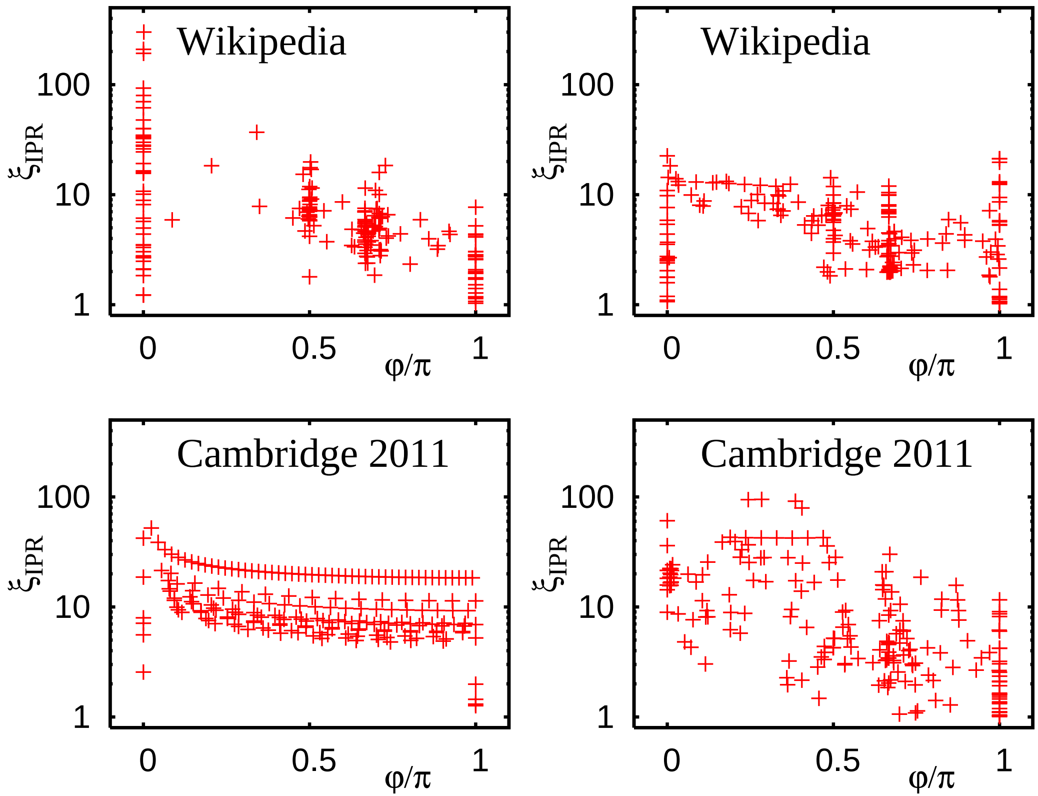
Power law decay of eigenvectors:



$$|\psi_i(K_i)| \sim K_i^b \quad \text{for} \quad K_i \geq 10^4$$

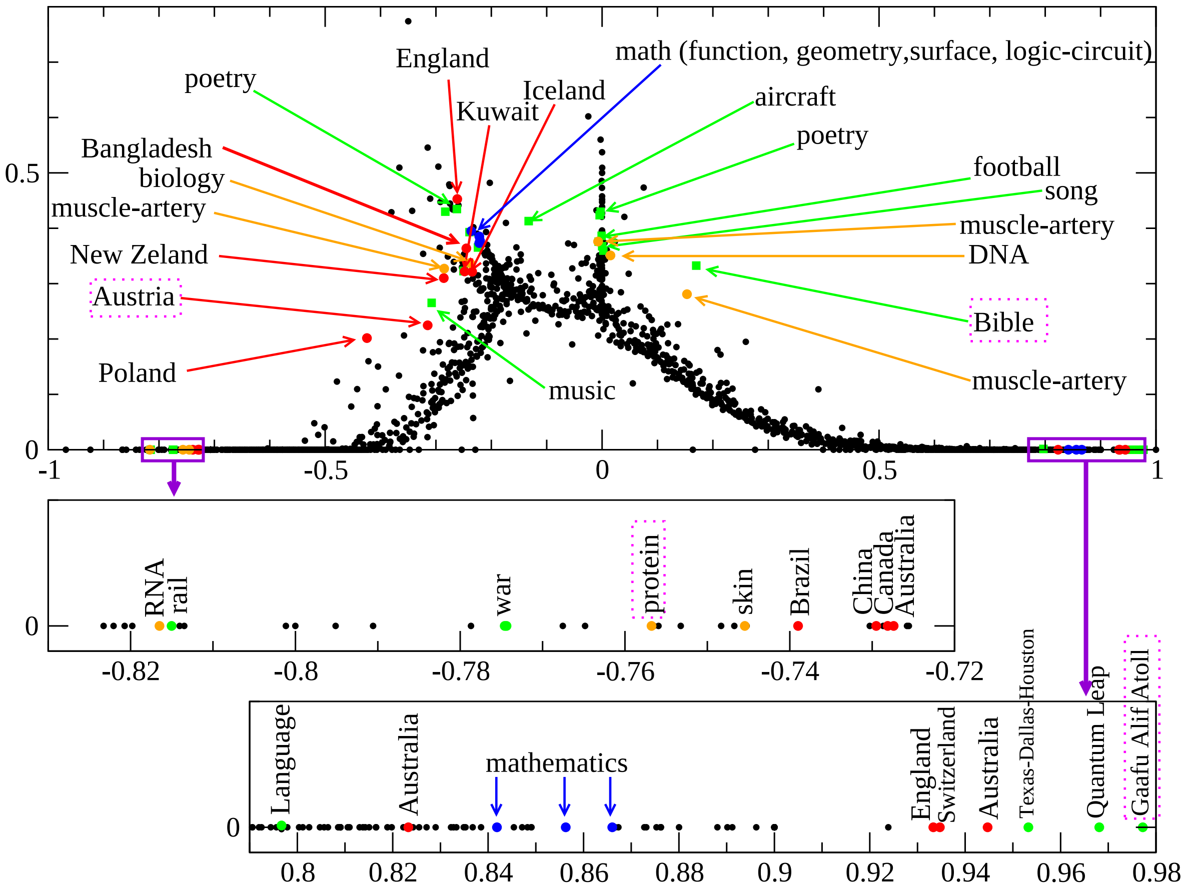
$$\varphi = \arg(\lambda_i)$$

Inverse participation ratio of eigenvectors:

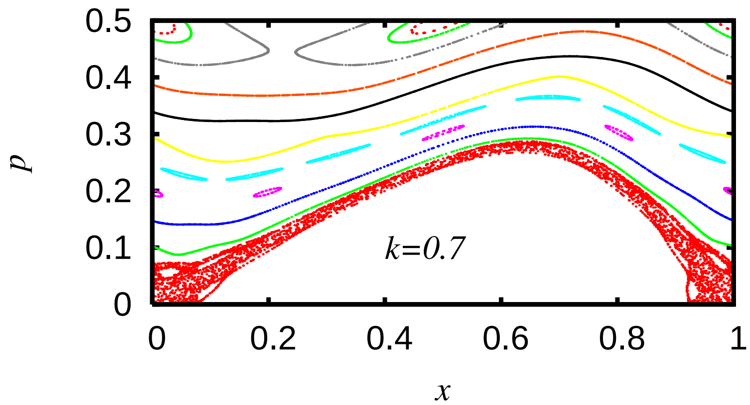


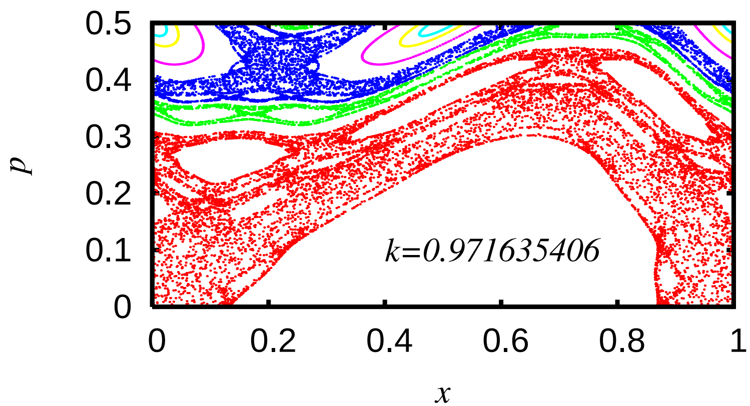
$$\xi_{\rm IPR} = (\sum_j |\psi_i(j)|^2)^2 / \sum_j |\psi_i(j)|^4$$
$$\varphi = \arg(\lambda_i)$$

“Themes” of certain eigenvectors:



Chirikov Standard map





$$p_{n+1} = p_n + \frac{k}{2\pi} \sin(2\pi\, x_n)$$

$$x_{n+1} = x_n + p_{n+1}$$

x and p are taken modulo 1 and the symmetry $(x, p) \to (1 - x, 1 - p)$ allows to restrict: $x \in [0,\ 1]$ and $p \in [0,\ 0.5]$.

Transition to "global" chaos at $k_c = 0.971635406$.

Perron-Frobenius matrix for chaotic maps

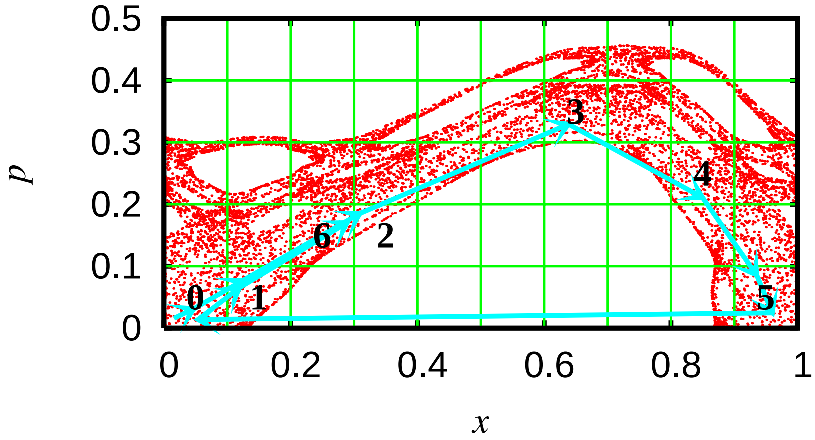
A new variant of the ***Ulam Method*** to construct the ***Perron-Frobenius matrix*** for the case of a mixed phase space:

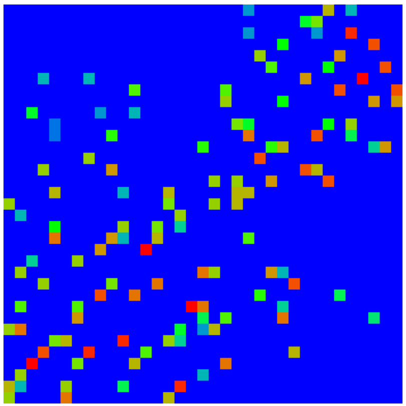
- Subdivide x space in M cells and p space in $M/2$ cells with M being an (even) integer number.
- Iterate (for a very long time: $t \sim 10^{11} - 10^{12}$) a classical trajectory and attribute a new number to each new cell which is entered. At the same time count the number of transitions from cell i to cell j ($\Rightarrow\ n_{ji}$).
- Calculate the $N \times N$ matrix

$$G_{ji} = \frac{n_{ji}}{\sum_l n_{li}}$$

of dimension $N \approx M^2/4$ and which is a (sparse) Perron Frobenius operator, i. e.: $G_{ji} \geq 0$, $\sum_j G_{ji} = 1$, G_{ji} sparse.

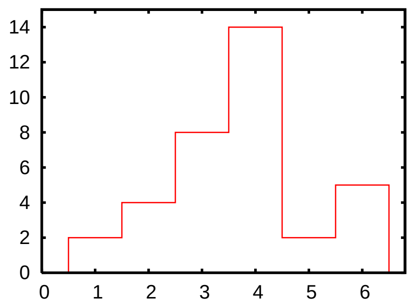
$M = 10, t = 10^6 \quad \Rightarrow \quad N = 35$





density plot of matrix elements

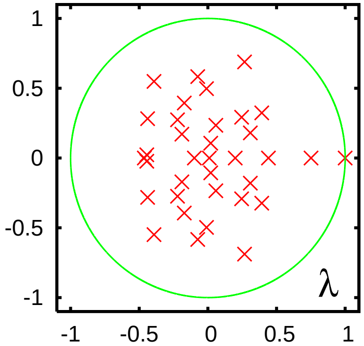
(blue=min=0, green=medium, red=max)

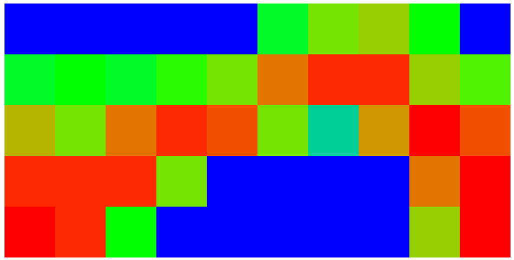


distribution of number of non-zero matrix elements per column

Eigenvalues

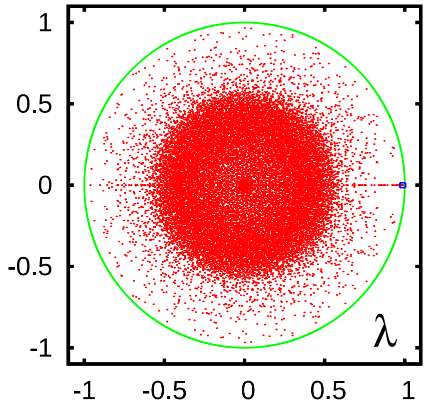
for $M = 10$, $t = 10^6$ and $N = 35$

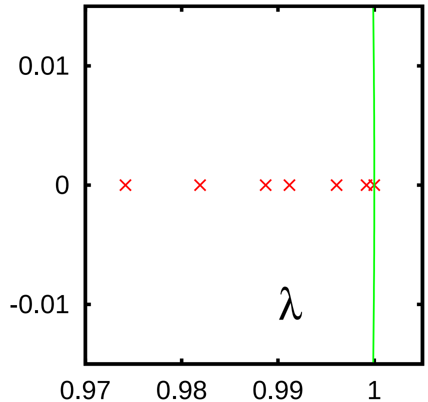




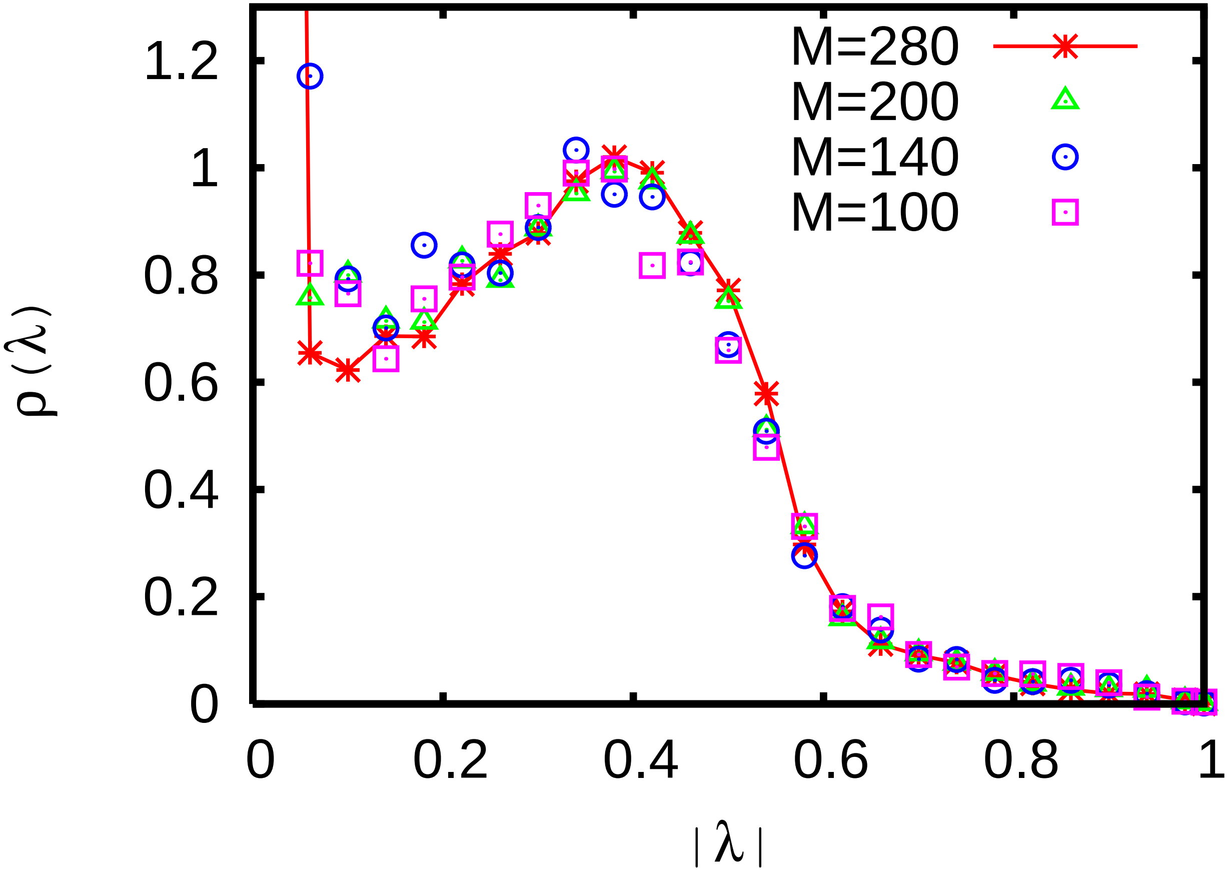
Phase space representation of the eigenvector for $\lambda_0 = 1$.

for $M = 280$, $t = 10^{12}$ and $N = 16609$

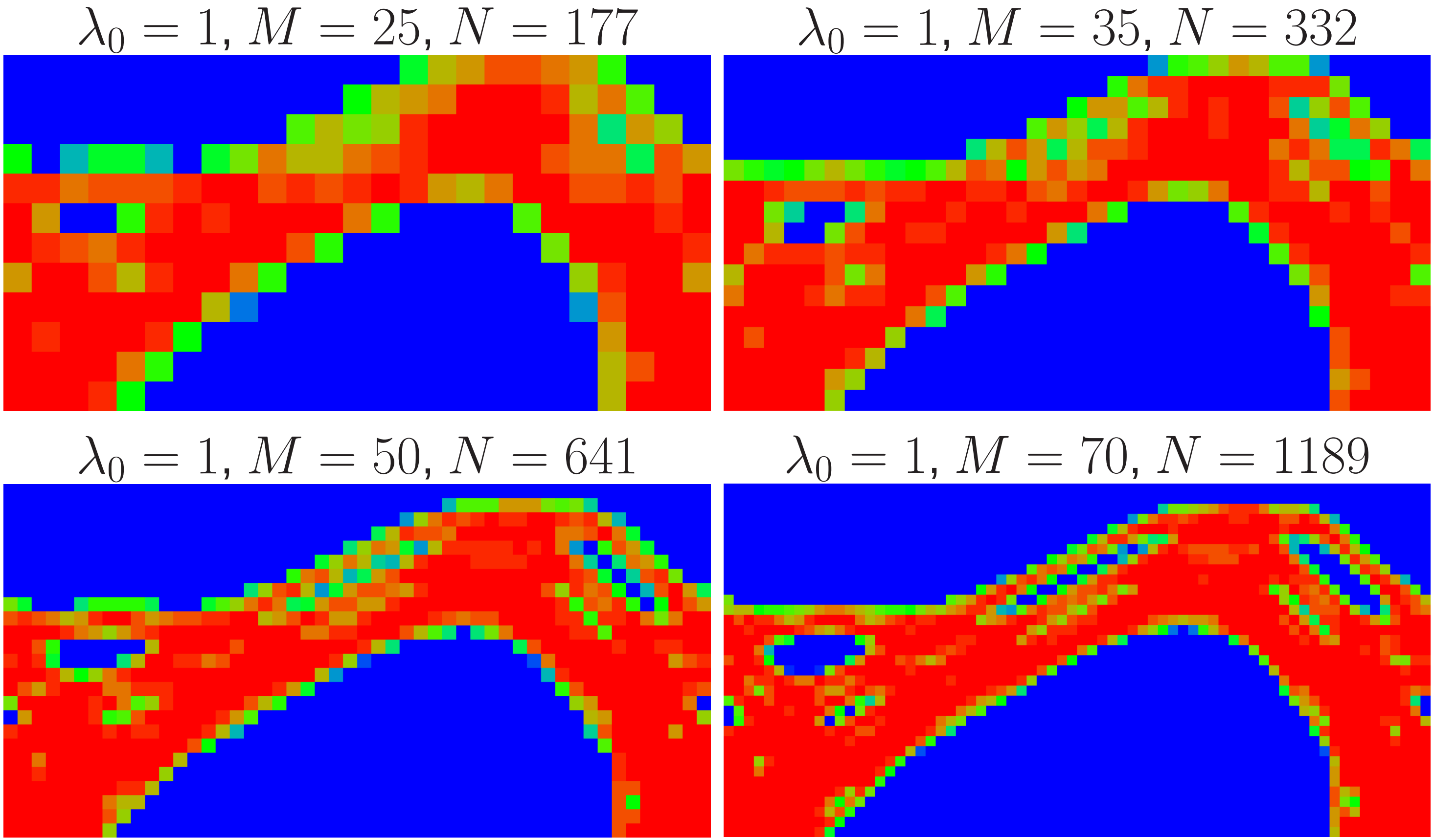


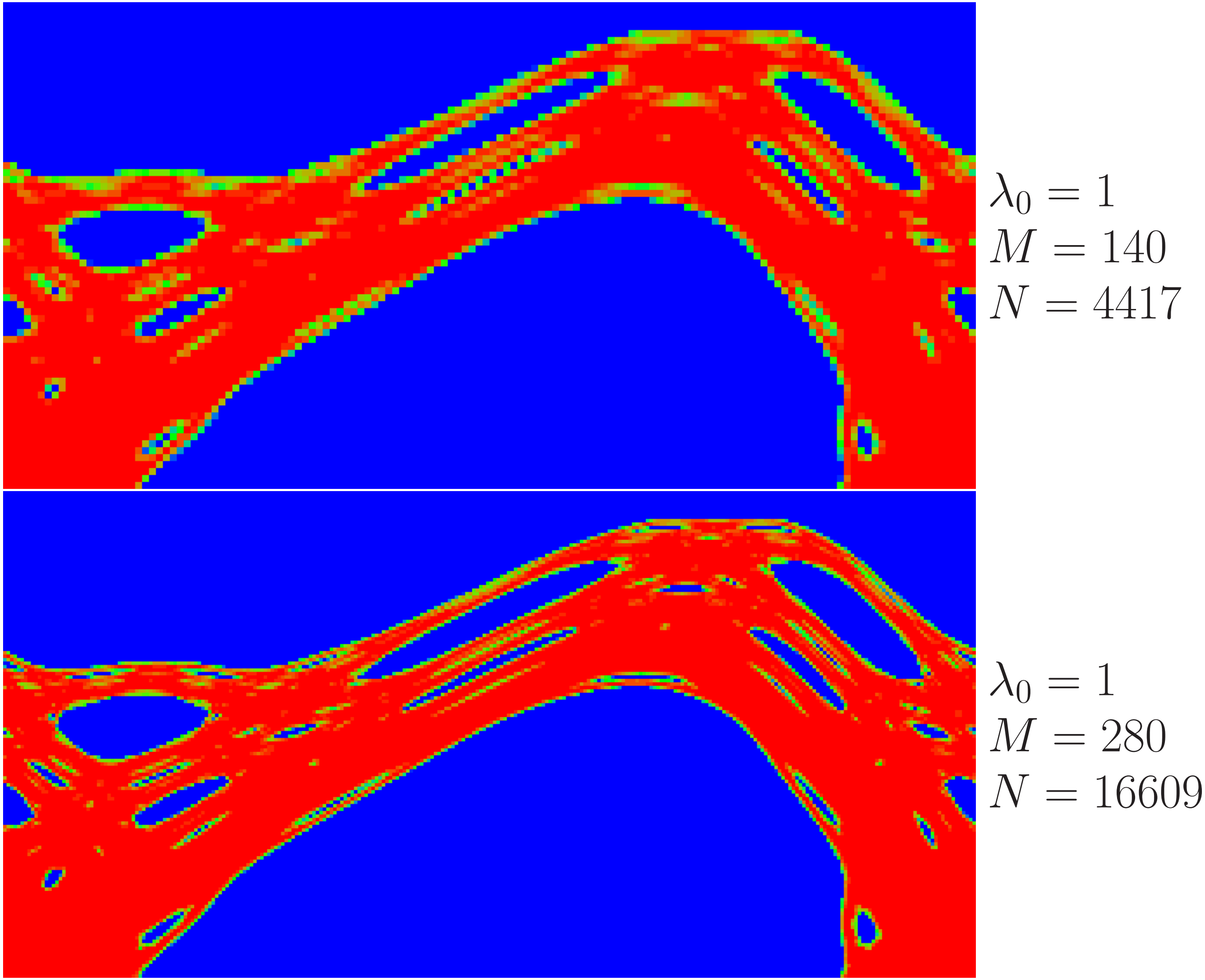


Complex density of states



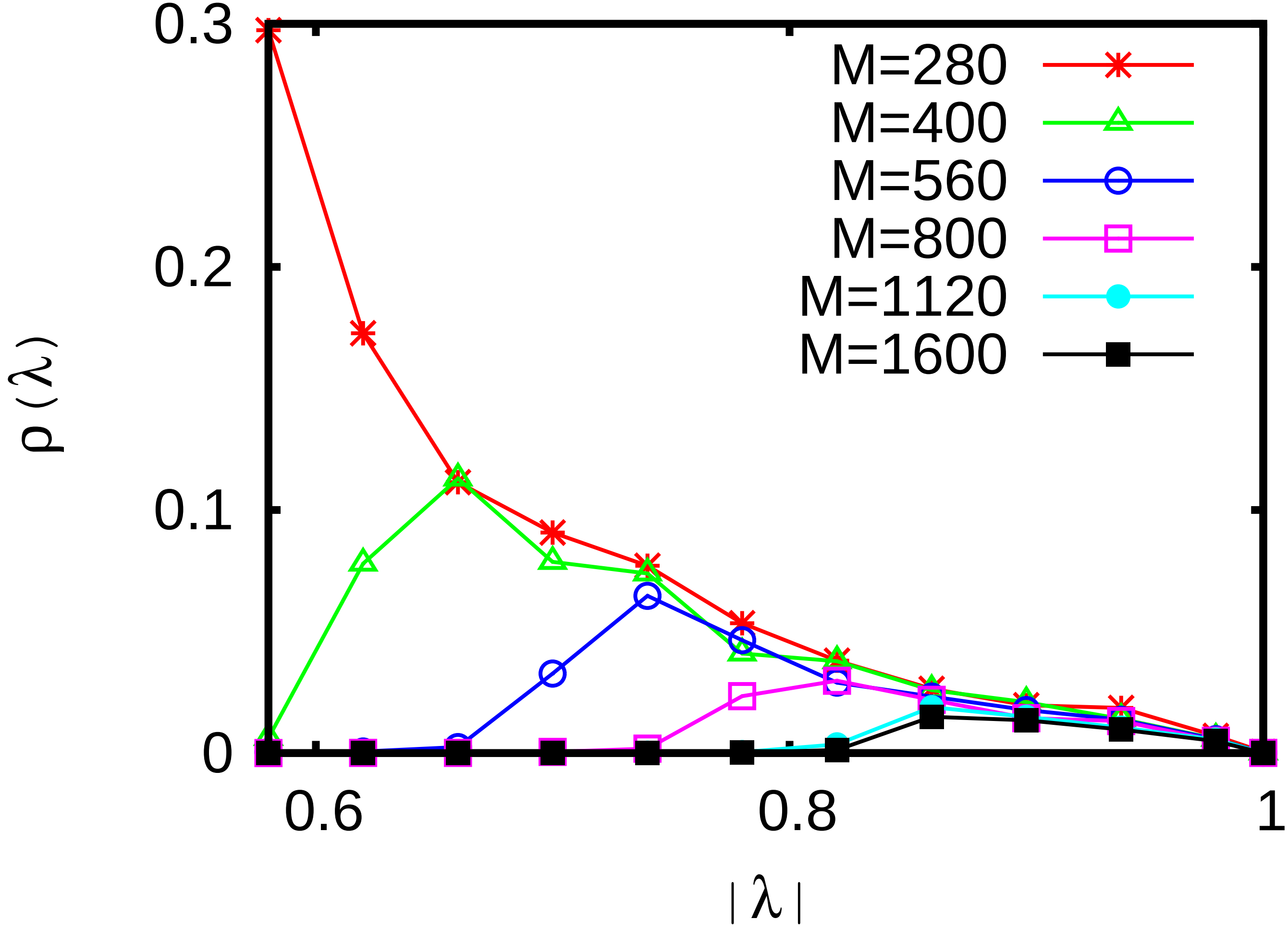
Eigenvectors

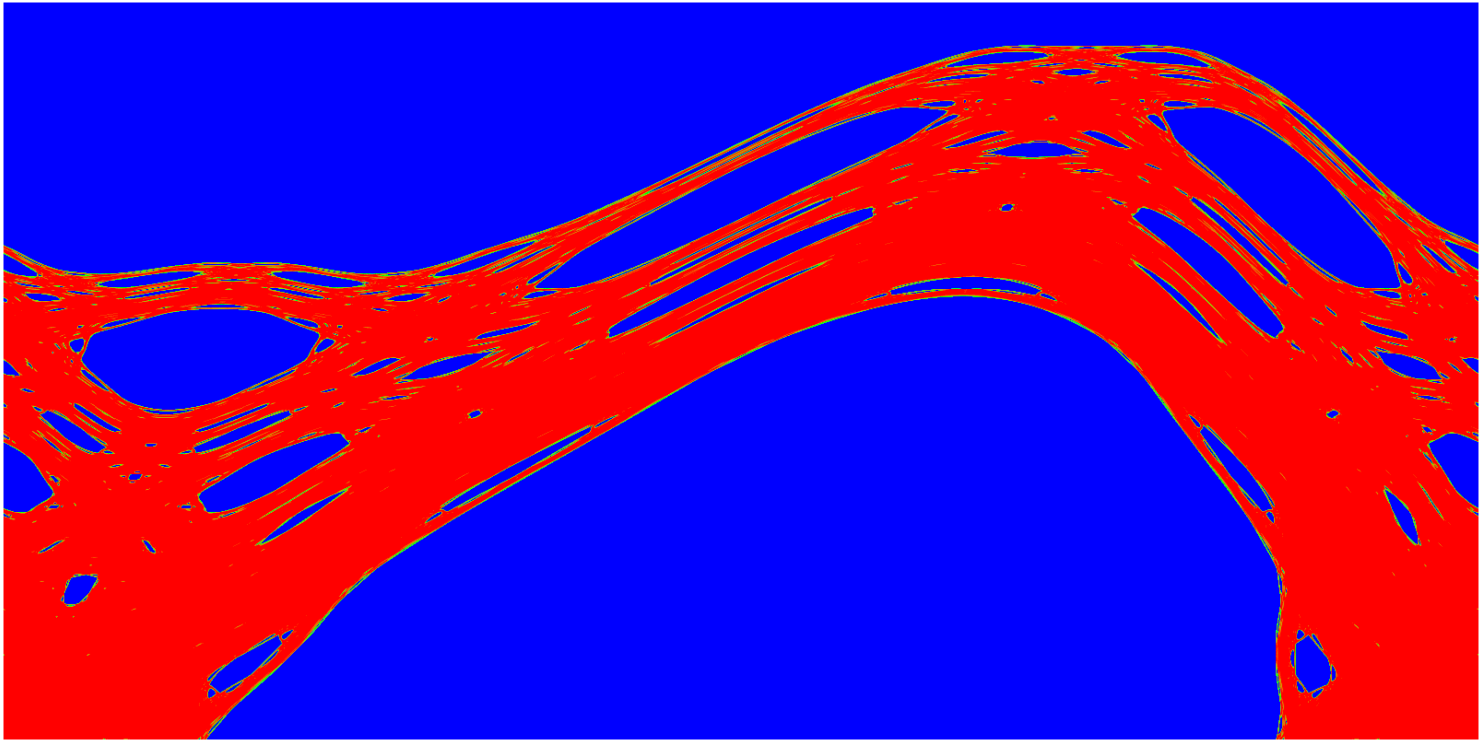




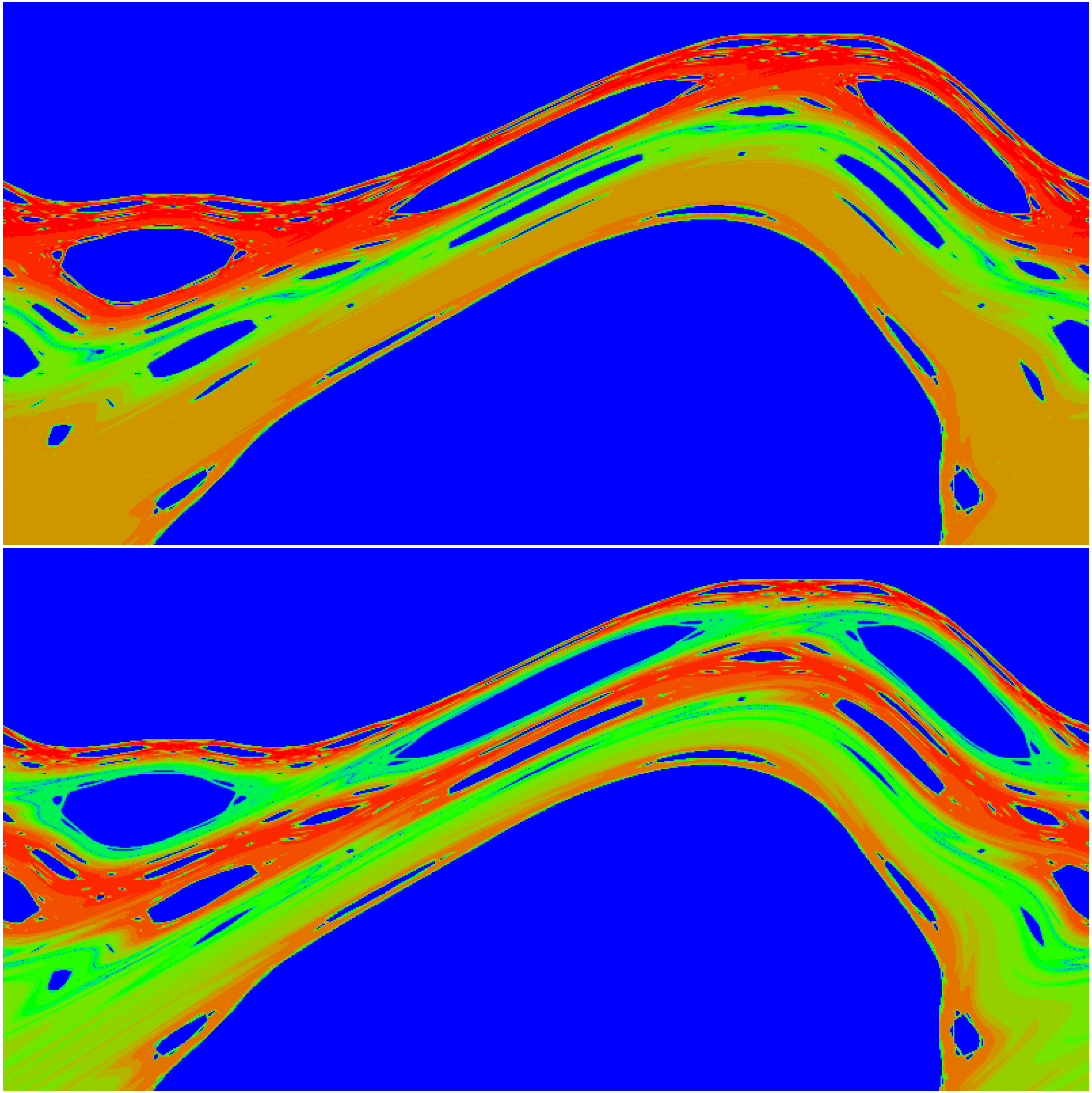
$\lambda_0 = 1$
$M = 140$
$N = 4417$
$\lambda_0 = 1$
$M = 280$
$N = 16609$

complex density of states:





$\lambda_0 = 1$, $M = 1600$, $N = 494964$, $n_A = 3000$



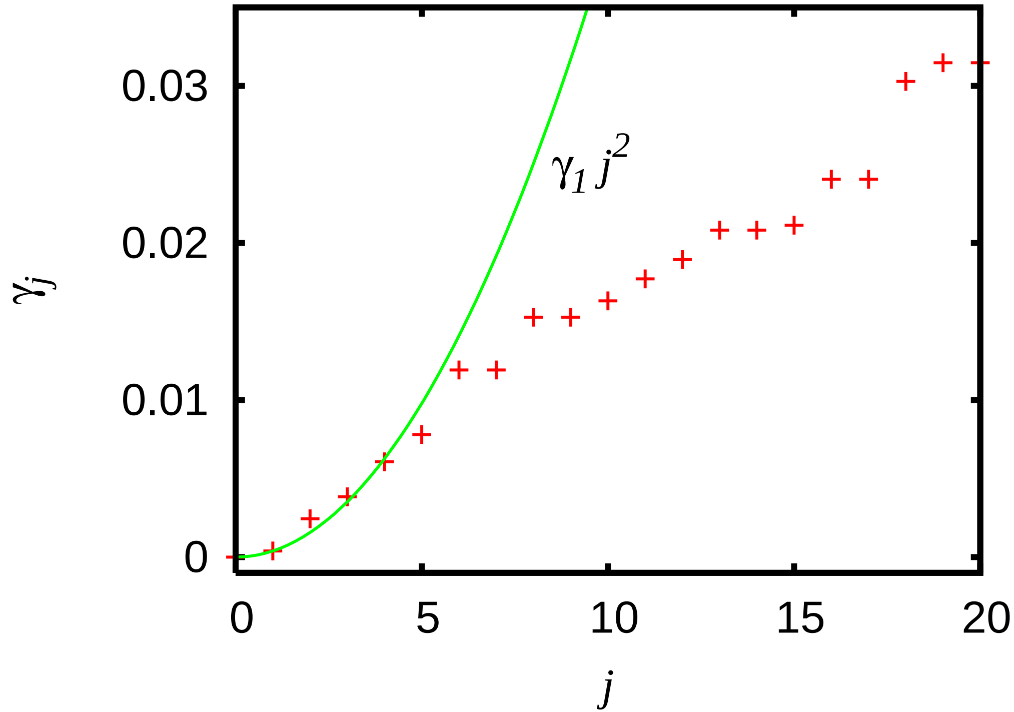
$\lambda_1 =$
0.99980431

$M = 800$
$N = 127282$
$n_A = 2000$

$\lambda_2 =$
0.99878108

$M = 800$
$N = 127282$
$n_A = 2000$

Diffuson modes

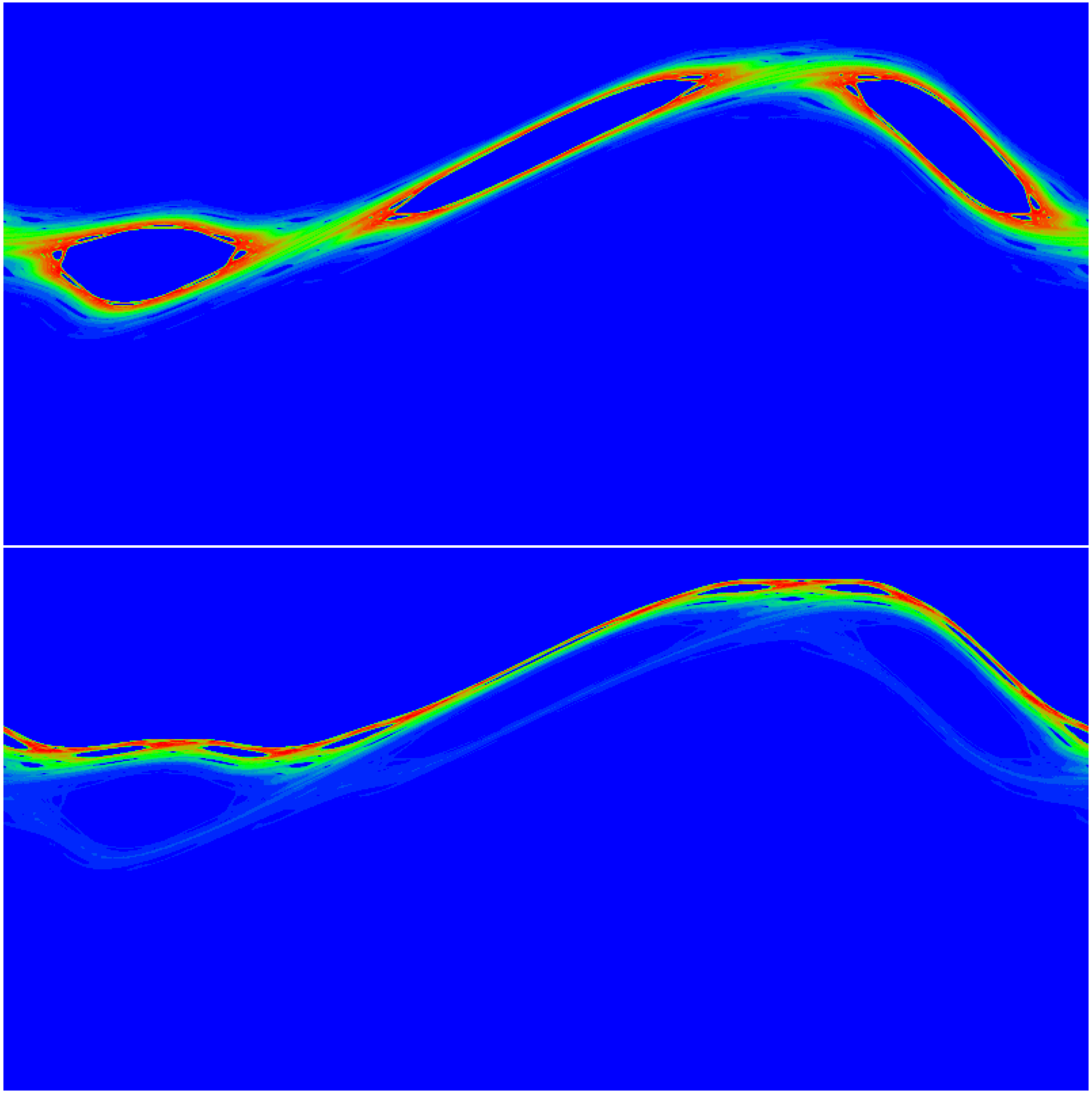


$$\gamma_j = -2\ln(|\lambda_j|)$$

$$\gamma_j \approx \gamma_1 j^2$$

for $j \leq 5$.

What about eigenvectors for complex or real negative λ_j ?

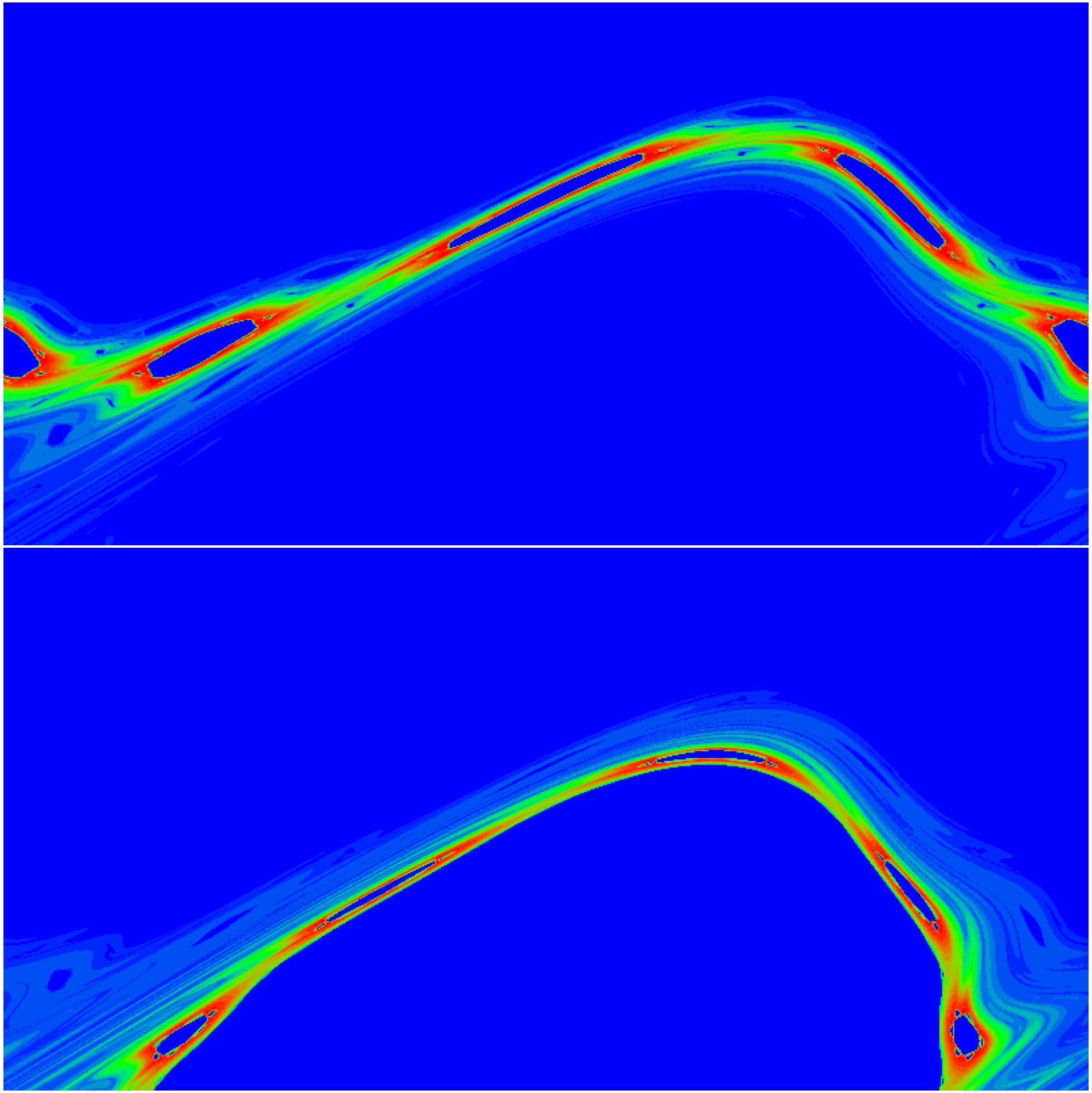


$\lambda_6 =$
-0.49699831
$+i\,0.86089756$
$\approx |\lambda_6|\, e^{i\,2\pi/3}$

$M = 800$
$N = 127282$
$n_A = 2000$

$\lambda_{19} =$
-0.71213331
$+i\,0.67961609$
$\approx |\lambda_{19}|\, e^{i\,2\pi(3/8)}$

$M = 800$
$N = 127282$
$n_A = 2000$



$\lambda_8 =$
0.00024596
$+i\,0.99239222$
$\approx |\lambda_8|\, e^{i\,2\pi/4}$

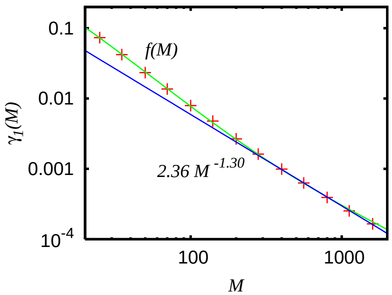
$M = 800$
$N = 127282$
$n_A = 2000$

$\lambda_{13} =$
0.30580631
$+i\,0.94120900$
$\approx |\lambda_{13}|\, e^{i\,2\pi/5}$

$M = 800$
$N = 127282$
$n_A = 2000$

Extrapolation of eigenvalues

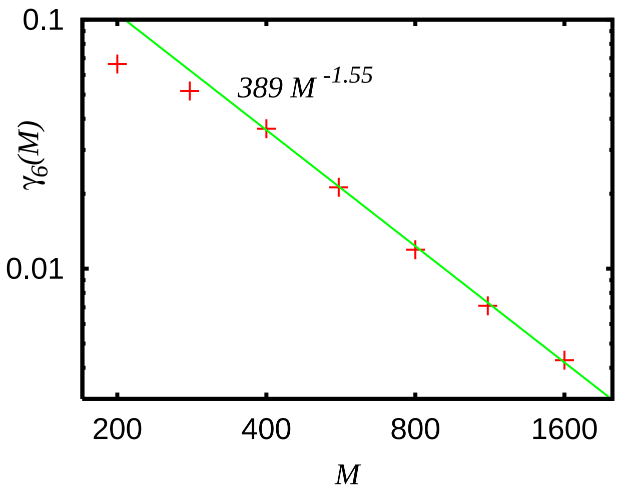
$\gamma_1(M)$ in the limit $M \to \infty$:

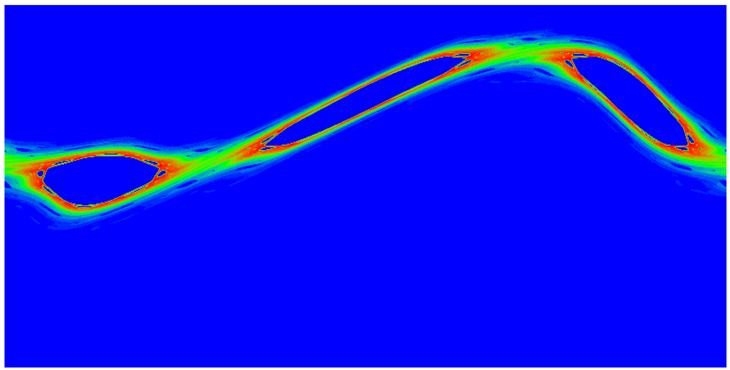


$$f(M) = \frac{D}{M} \frac{1 + \frac{C}{M}}{1 + \frac{B}{M}}$$

$$D = 0.245$$
$$B = 13.1$$
$$C = 258$$

$\gamma_6(M)$ in the limit $M \to \infty$:

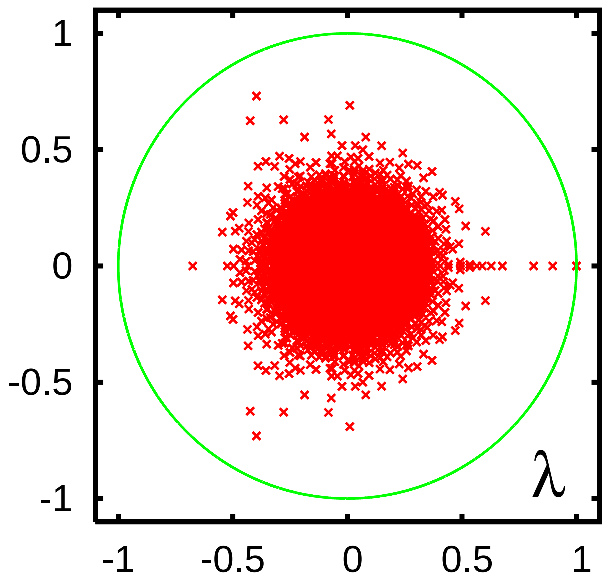


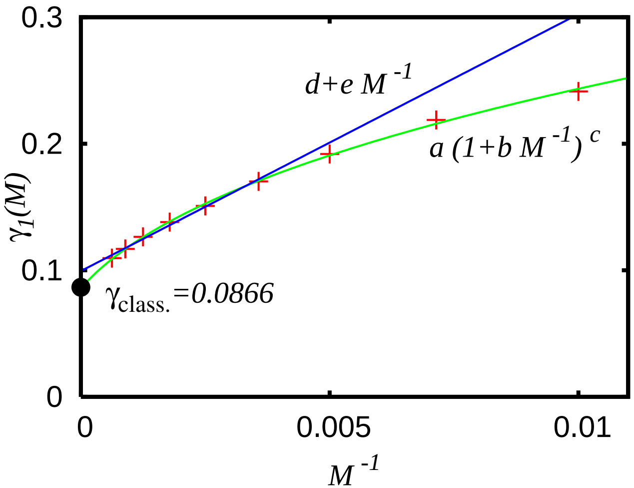


$\gamma_6(M) \approx 389\, M^{-1.55}$ for $M \geq 400$.

Strong chaos

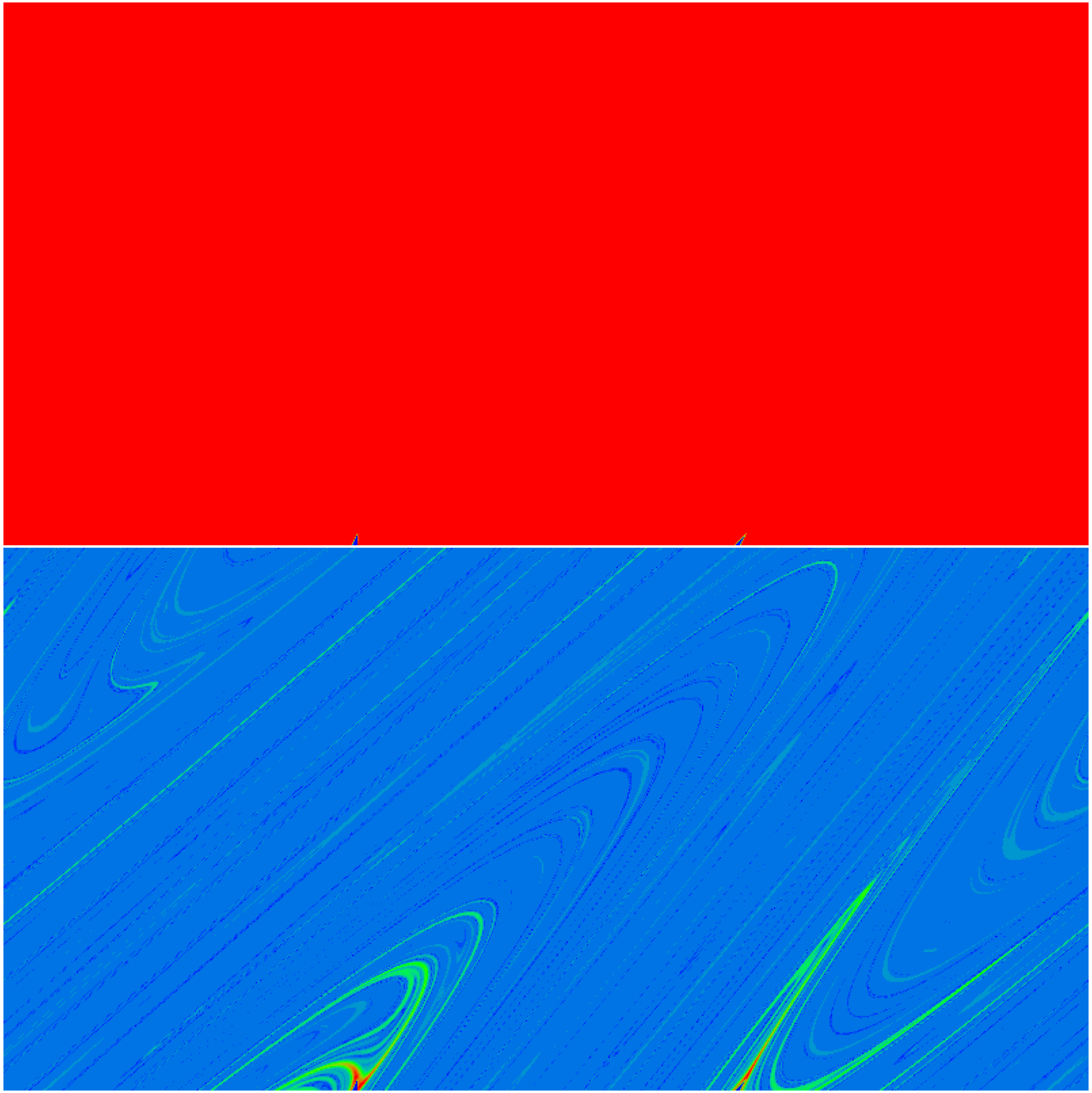
at $k = 7$, $M = 140$, $t = 10^{11}$ and $N = 9800$





$$a \approx 0.0857 \pm 0.0036,\ d \approx 0.0994$$

$$\lim_{M\to\infty} \gamma_1(M) > 0$$

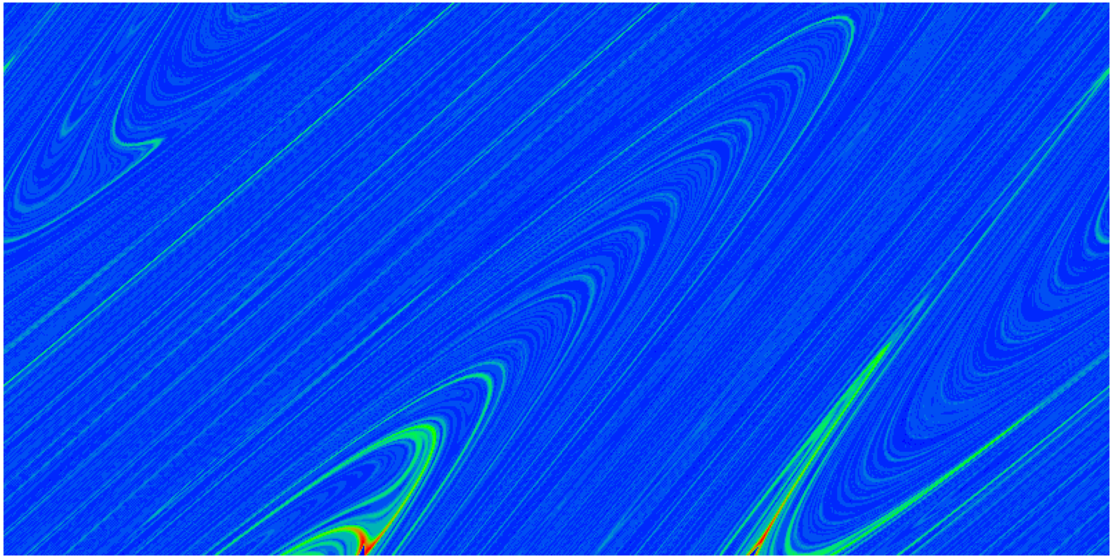


$\lambda_0 = 1$

$k = 7$
$M = 800$
$N = 319978$
$n_A = 1500$

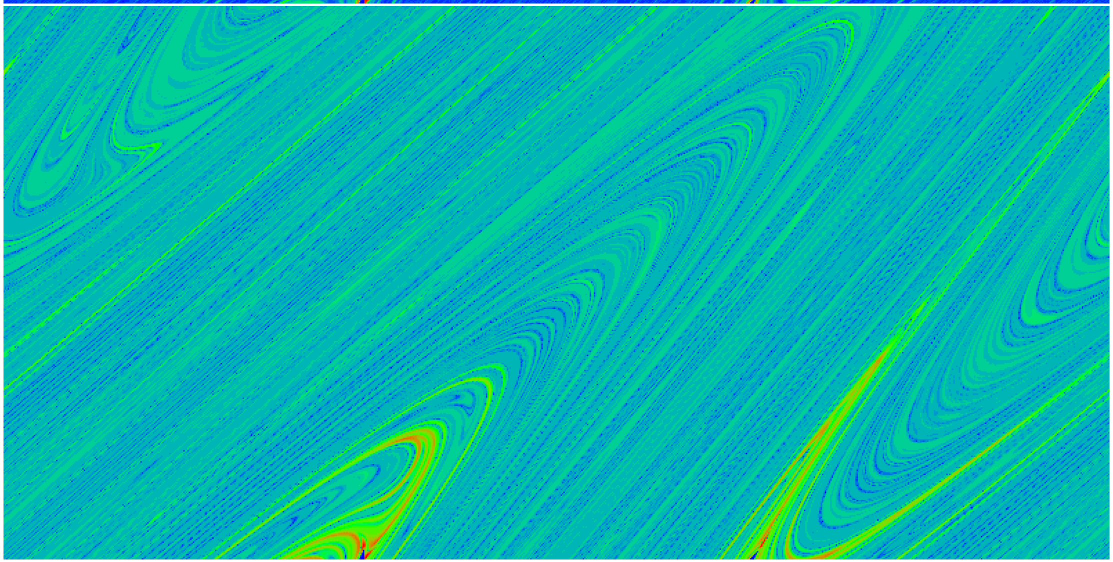
$\lambda_1 =$
0.93874817

$k = 7$
$M = 800$
$N = 319978$
$n_A = 1500$



$\lambda_2 =$
-0.49264273
$+i0.78912368$

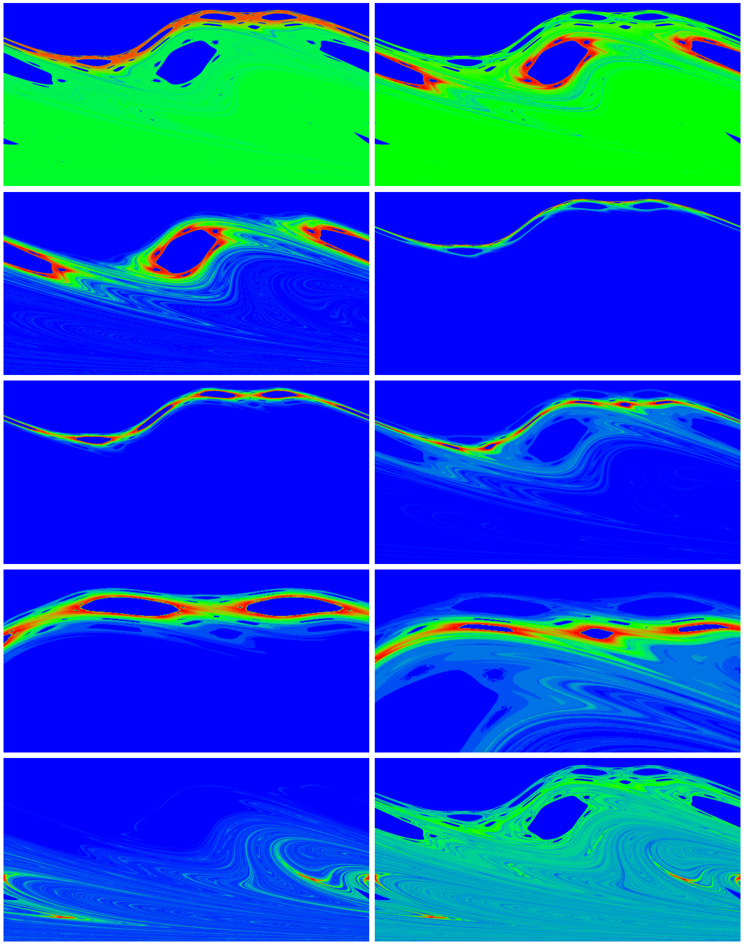
$k = 7$
$M = 800$
$N = 319978$
$n_A = 1500$



$\lambda_8 =$
0.87305253

$k = 7$
$M = 800$
$N = 319978$
$n_A = 1500$

Separatrix map

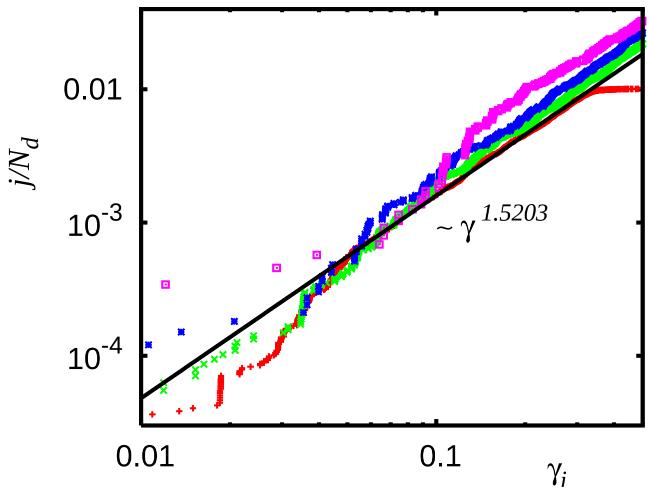


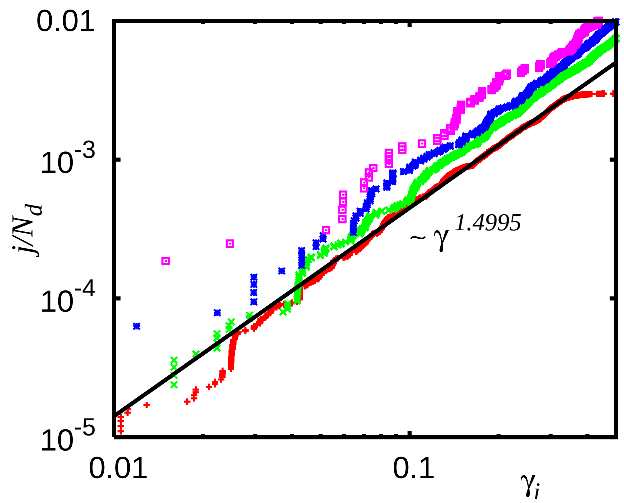
$$\bar{p} = p + \sin(2\pi x)$$

$$\bar{x} = x + \frac{\Lambda}{2\pi}\ln(|\bar{p}|) \pmod 1$$

$$\Lambda = \Lambda_c = 3.1819316$$

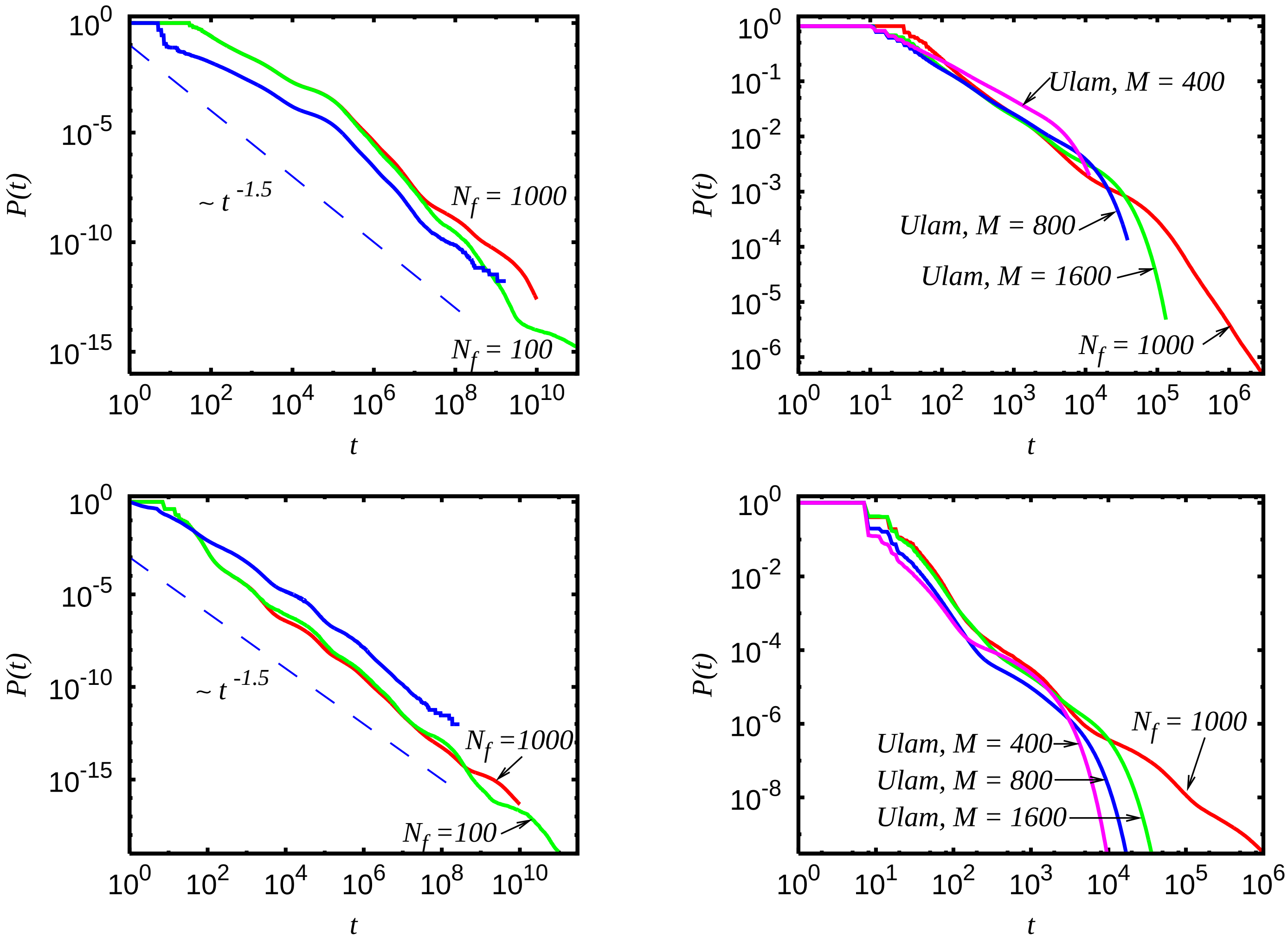
Poincaré recurrences



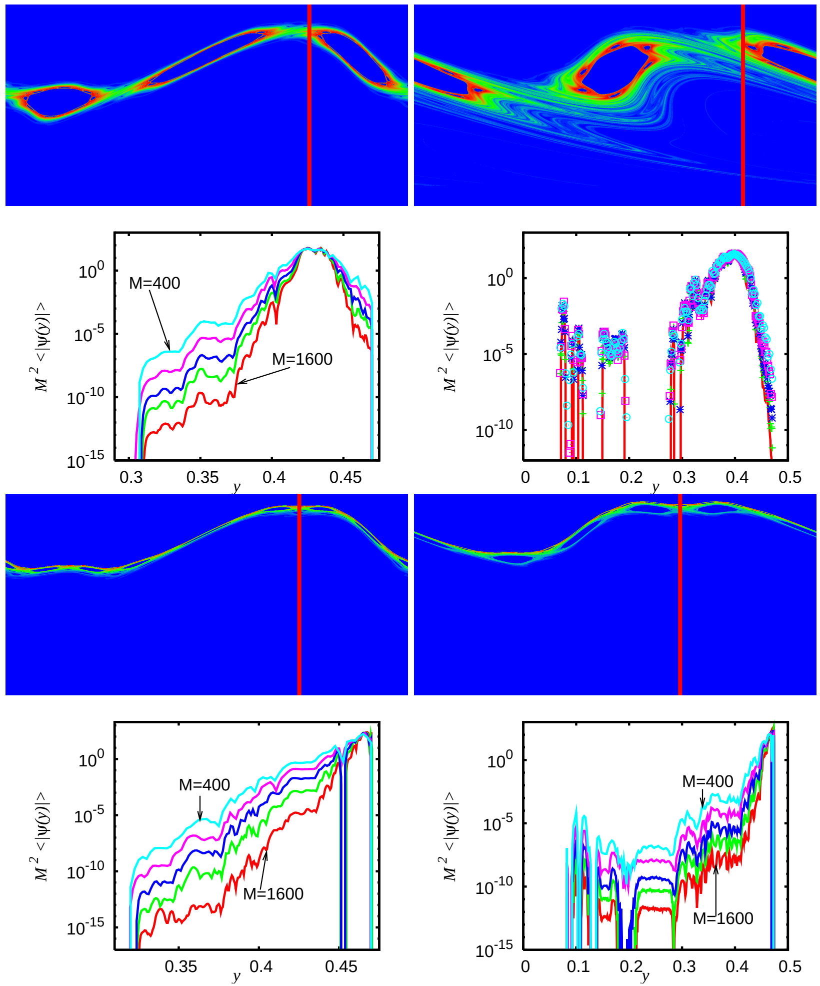


$$\rho_\Sigma(\gamma) \sim \gamma^\beta \;,\; \beta \approx 1.5 \;.$$

$$P(t) \sim \int_0^1 \frac{d\rho_\Sigma(\gamma)}{d\gamma} \exp(-\gamma t)\, d\gamma \;\sim\; \frac{1}{t^\beta}$$


P(t)
t
~ t -1.5
N_f = 1000
N_f = 100
Ulam, M = 400
Ulam, M = 800
Ulam, M = 1600
N_f = 1000
N_f =1000
N_f =100
Ulam, M = 400
Ulam, M = 800
Ulam, M = 1600
N_f = 1000

Phase space localization



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3. K. M. Frahm and D. L. Shepelyansky, ***Ulam method for the Chirikov standard map***, Eur. Phys. J. B **76**, 57 (2010).

4. L. Ermann, K. M. Frahm and D. L. Shepelyansky, **Spectral properties of Google matrix of Wikipedia and other networks**, Eur. Phys. J. B **86**, 193 (2013).

5. K. M. Frahm and D. L. Shepelyansky, **Poincaré recurrences and Ulam method for the Chirikov standard map**, Eur. Phys. J. B **86**, 322 (2013).