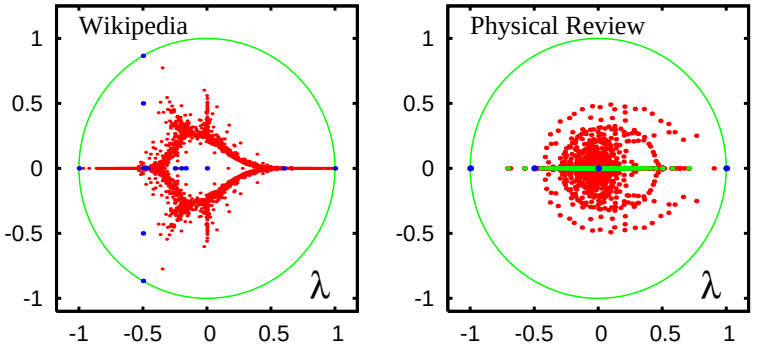

Spectral properties of Google matrix

Lecture 3

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Network analysis and applications Luchon, June 21 - July 5, 2014

Contents

Random Perron-Frobenius matrices

Construct random matrix ensembles G_{ij} such that:

- $G_{ij} \geq 0$
- G_{ij} are (approximately) non-correlated and distributed with the same distribution $P(G_{ij})$ (of finite variance σ^2).
- $\sum_j G_{ij} = 1 \quad \Rightarrow \quad \langle G_{ij} \rangle = 1/N$
- $\Rightarrow$ average of G has one eigenvalue $\lambda_1 = 1$ ($\Rightarrow$ "flat" PageRank) and other eigenvalues $\lambda_j = 0$ (for $j \neq 1$).
- degenerate perturbation theory for the fluctuations $\Rightarrow$ circular eigenvalue density with $R = \sqrt{N}\sigma$ and one unit eigenvalue.

Different variants of the model:

- ***uniform full***: $P(G) = N/2$ for $0 \leq G \leq 2/N$

  $\Rightarrow \quad R = 1/\sqrt{3N}$

- ***uniform sparse*** with Q non-zero elements per column:
  $P(G) = Q/2$ for $0 \leq G \leq 2/Q$ with probability Q/N
  and $G = 0$ with probability $1 - Q/N$

  $\Rightarrow \quad R = 2/\sqrt{3Q}$

- ***constant sparse*** with Q non-zero elements per column:
  $G = 1/Q$ with probability Q/N
  and $G = 0$ with probability $1 - Q/N$

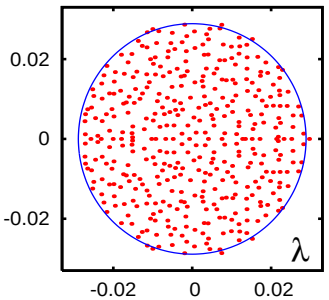
  $\Rightarrow \quad R = 1/\sqrt{Q}$

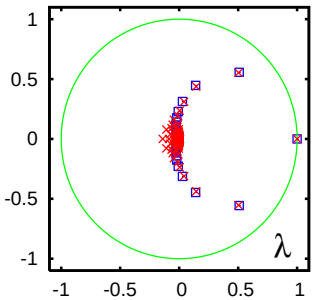
- ***powerlaw*** with $p(G) = D(1 + aG)^{-b}$ for $0 \leq G \leq 1$ and $2 < b < 3$:

  $\Rightarrow \quad R = C(b)\, N^{1-b/2} \quad , \quad C(b) = (b-2)^{(b-1)/2} \sqrt{\frac{b-1}{3-b}}$

Numerical verification:

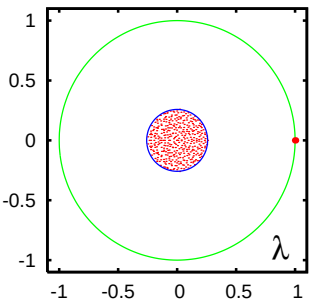
uniform full:
$N = 400$

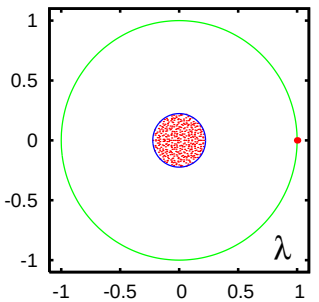




triangular random and average

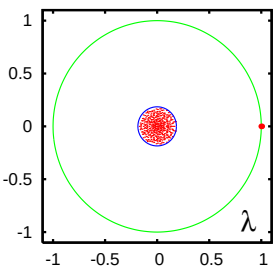
uniform sparse:
$N = 400,$
$Q = 20$

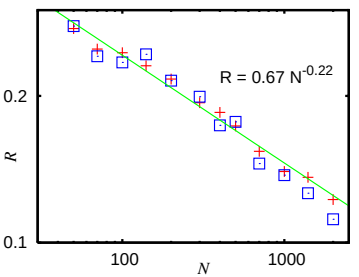




constant sparse:
$N = 400,$
$Q = 20$

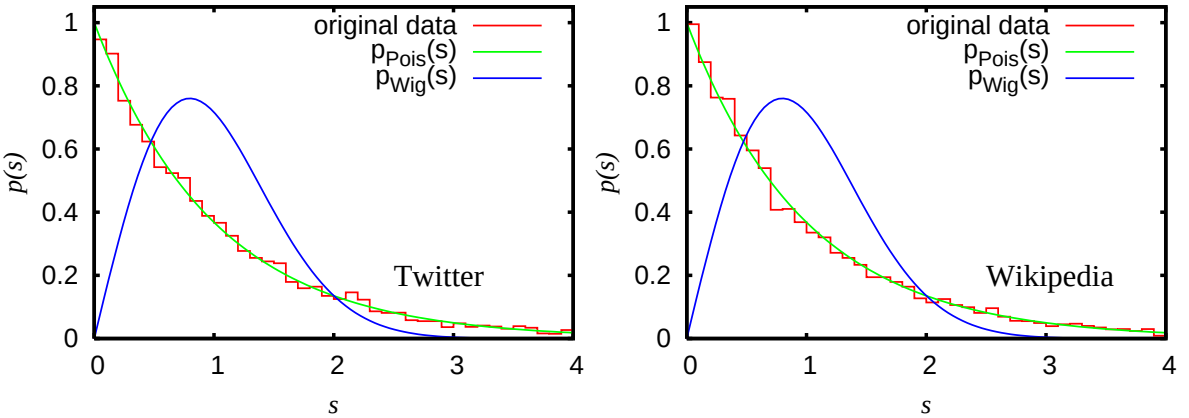
power law:
$b = 2.5$





power law case:
$R_{\mathrm{th}} \sim N^{-0.25}$

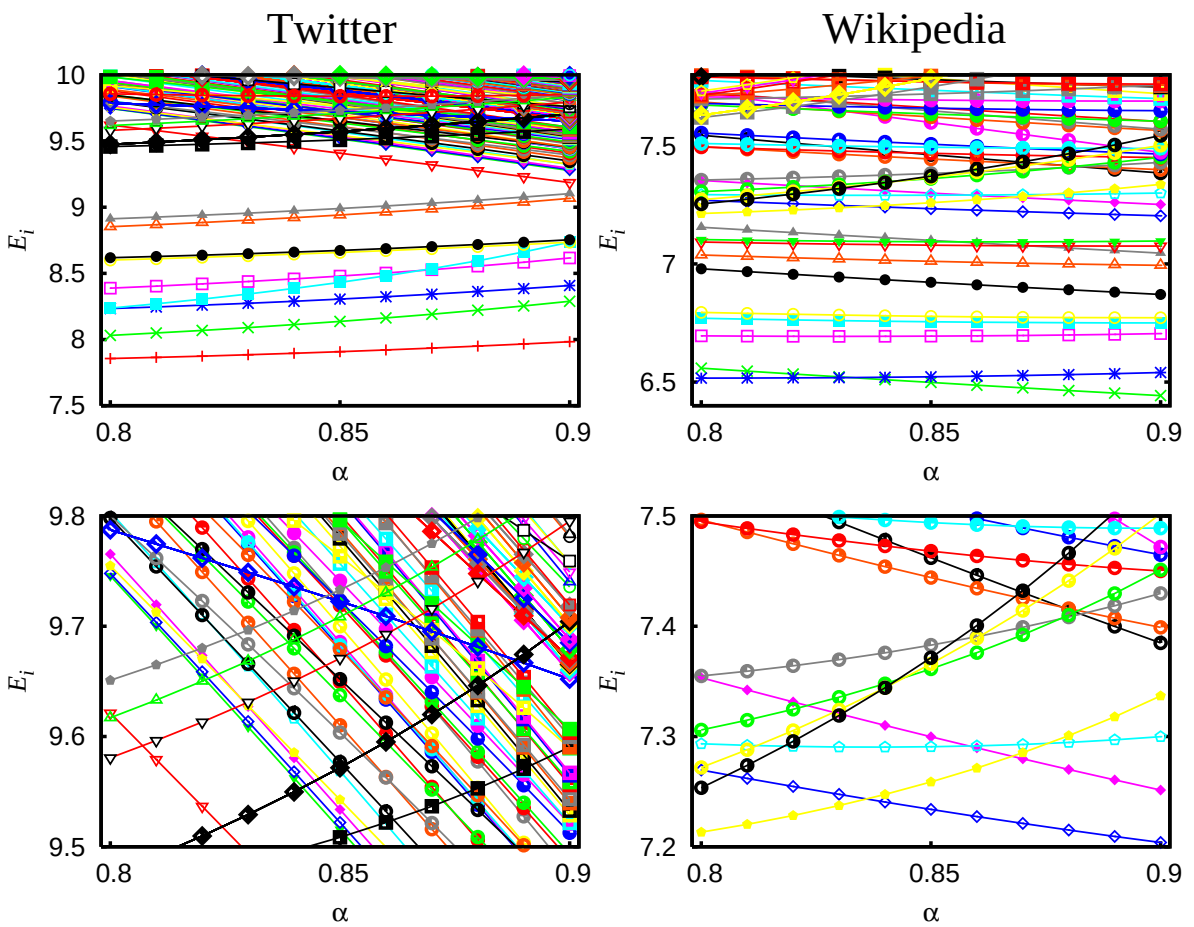
Poisson statistics of PageRank



Identify PageRank values to “energy-levels”:

$$P(i) = \exp(-E_i/T)/Z$$

with $Z = \sum_i \exp(-E_i/T)$ and an effective temperature T (can be choosen: $T = 1$).

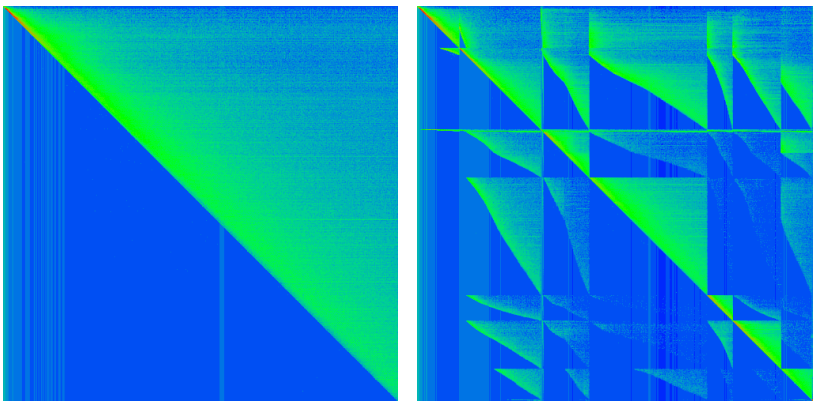


Parameter dependance of $E_i = -\ln(P_i)$ on the damping factor α.

Physical Review network

$N = 463347$ nodes and $N_\ell = 4691015$ links.

Coarse-grained matrix structure (500×500 cells):

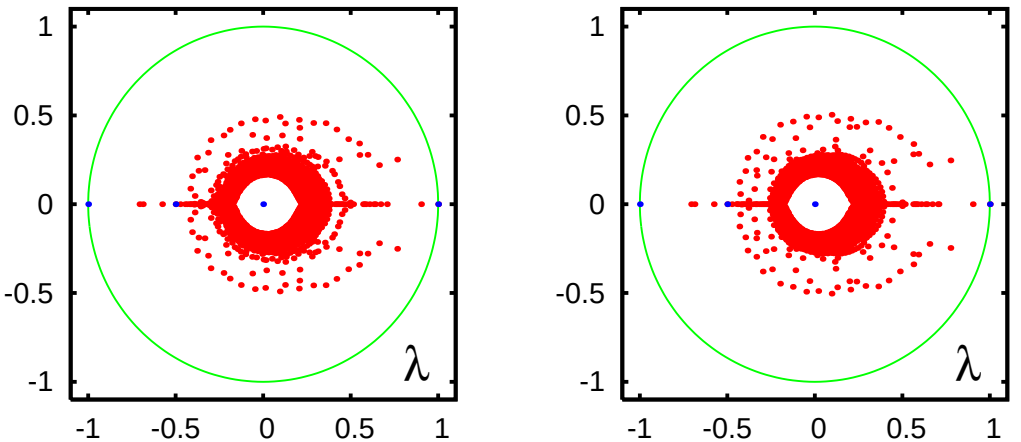


left: time ordered
right: journal and then time ordered

"11" Journals of Physical Review: (Phys. Rev. Series I), Phys. Rev., Phys. Rev. Lett., (Rev. Mod. Phys.), Phys. Rev. A, B, C, D, E, (Phys. Rev. STAB and Phys. Rev. STPER).

$\Rightarrow$ nearly triangular matrix structure of adjacency matrix: most citations links $t \to t'$ are for $t > t'$ ("past citations") but there is small number ($12126 = 2.6 \times 10^{-3} N_\ell$) of links $t \to t'$ with $t \leq t'$ corresponding to ***future citations***.

Spectrum by "double-precision" Arnoldi method with $n_A = 8000$:



Numerical problem: eigenvalues with $|\lambda| < 0.3 - 0.4$ are not reliable!

Reason: large Jordan subspaces associated to the eigenvalue $\lambda = 0$.

“very bad” Jordan perturbation theory:
Consider a “perturbed” Jordan block of size D:

$$\begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ \varepsilon & 0 & \cdots & 0 & 0 \end{pmatrix}$$

characteristic polynomial: $\lambda^D - (-1)^D \varepsilon$
$\varepsilon = 0 \quad \Rightarrow \quad \lambda = 0$
$\varepsilon \neq 0 \quad \Rightarrow \quad \lambda_j = -\varepsilon^{1/D} \exp(2\pi i j/D)$
for $D \approx 10^2$ and $\varepsilon = 10^{-16} \quad \Rightarrow$ “Jordan-cloud” of artifical eigenvalues due to rounding errors in the region $|\lambda| < 0.3 - 0.4$.

Triangular approximation

Remove the small number of links due to “future citations”.
Semi-analytical diagonalization is possible:

$$\boxed{S = S_0 + e\, d^T/N}$$

where $e_n = 1$ for all nodes n, $d_n = 1$ for dangling nodes n and $d_n = 0$ otherwise. S_0 is the pure link matrix which is ***nil-potent***:

$$\boxed{S_0^l = 0} \text{ with } l = 352.$$

Let ψ be an eigenvector of S with eigenvalue λ and $C = d^T\psi$.

- If $C = 0 \;\Rightarrow\; \psi$ eigenvector of $S_0 \;\Rightarrow\; \lambda = 0$ since S_0 nil-potent.

  These eigenvectors belong to large Jordan blocks and are responsible for the numerical problems.

  Note: Similar situation as in ***network of integer numbers*** where $l = [\log_2(N)]$ and numerical instability for $|\lambda| < 0.01$.

- If $C \neq 0 \;\Rightarrow\; \lambda \neq 0$ since the equation $S_0\psi = -C\, e/N$ does not have a solution $\Rightarrow\; \lambda \mathbf{1} - S_0$ invertible.

$$\Rightarrow\; \psi = C\,(\lambda \mathbf{1} - S_0)^{-1}\, e/N = \frac{C}{\lambda} \sum_{j=0}^{l-1} \left(\frac{S_0}{\lambda}\right)^j e/N \quad .$$

$$\text{From } \lambda^l = (d^T\psi/C)\lambda^l \;\Rightarrow\; \boxed{\mathcal{P}_r(\lambda) = 0}$$

with the ***reduced polynomial*** of degree $l = 352$:

$$\mathcal{P}_r(\lambda) = \lambda^l - \sum_{j=0}^{l-1} \lambda^{l-1-j}\, c_j = 0 \quad , \quad c_j = d^T\, S_0^j\, e/N \; .$$

$\Rightarrow$ at most $l = 352$ eigenvalues $\lambda \neq 0$ which can be numerically determined as the zeros of $\mathcal{P}_r(\lambda)$.

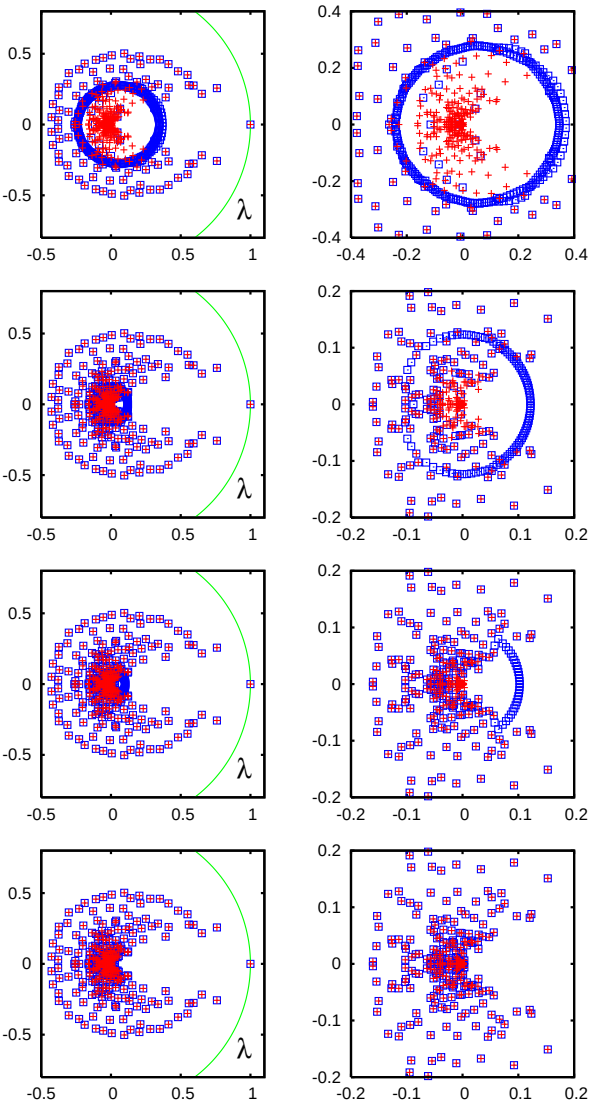
However: still numerical problems:

- $c_{l-1} \approx 3.6 \times 10^{-352}$
- alternate sign problem with a strong loss of significance.
- big sensitivity of eigenvalues on c_j

Solution:

Using the multi precision library GMP with 256 binary digits the zeros of $\mathcal{P}_r(\lambda)$ can be determined with accuracy $\sim 10^{-18}$.

Furthermore the Arnoldi method can also be implemented with higher precision.

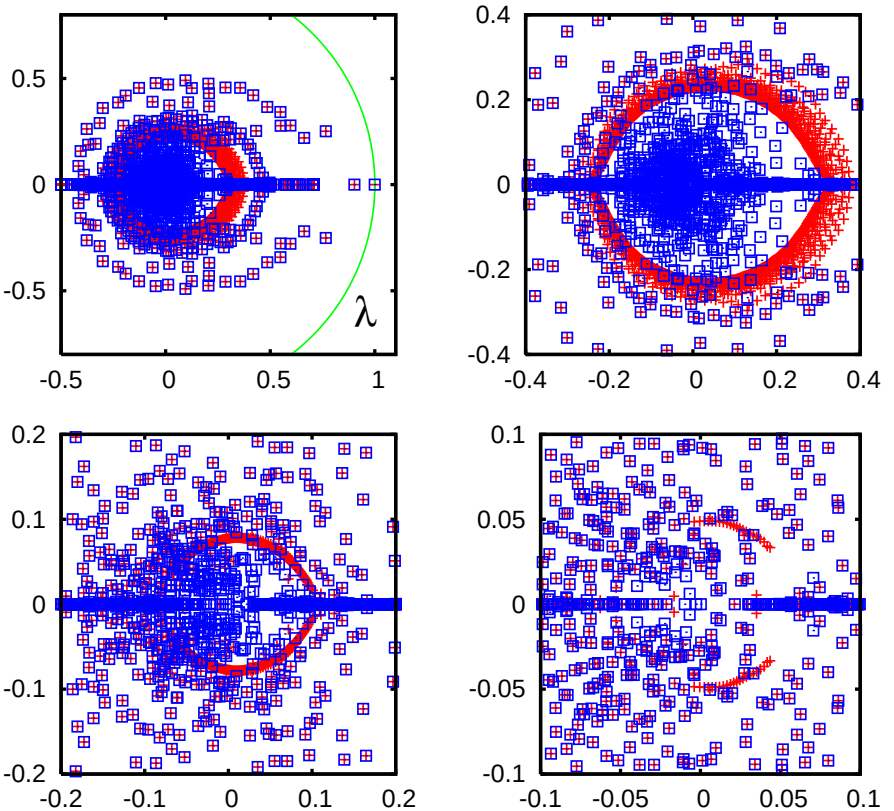


red crosses: zeros of $\mathcal{P}_r(\lambda)$ from 256 binary digits calculation

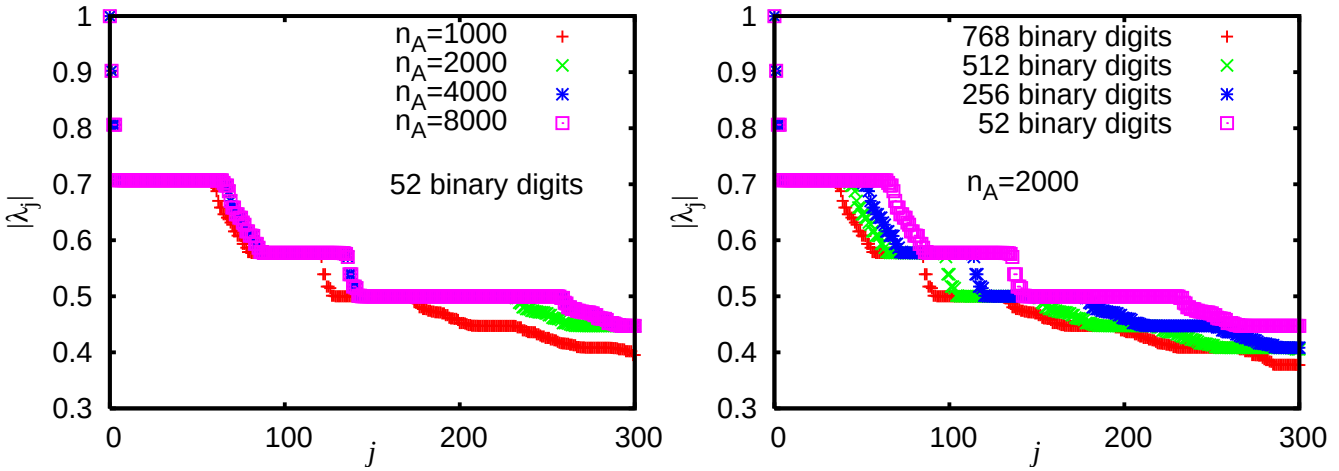
blue squares: eigenvalues from Arnoldi method with 52, 256, 512, 1280 binary digits. In the last case: $\Rightarrow$ break off at $n_A = 352$ with vanishing coupling element.

Full Physical Review network

High precision Arnoldi method for full Physical Review network (including the “future citations”) for 52, 256, 512, 768 binary digits and $n_A = 2000$:



Degeneracies



High precision in Arnoldi method is “bad” to count the degeneracy of certain degenerate eigenvalues.

In theory the Arnoldi method cannot find several eigenvectors for degenerate eigenvalues, a shortcoming which is (partly) “repaired” by rounding errors.

Q: How are highly degenerate core space eigenvalues possible ?

Semi-analytical argument for the full PR network:

$$\boxed{S = S_0 + e\, d^T/N}$$

There are ***two groups of eigenvectors*** ψ with: $S\psi = \lambda\psi$

1. Those with $d^T\psi = 0 \quad \Rightarrow \quad \psi$ is also an eigenvector of S_0. Generically an arbitrary eigenvector of S_0 is **not** an eigenvector of S **unless** the eigenvalue is degenerate with degeneracy $m > 1$. Using linear combinations of different eigenvectors for the same eigenvalue one can construct $m - 1$ eigenvectors ψ respecting $d^T\psi = 0$ which are therefore eigenvectors of S.

   Pratically: determine degenerate subspace eigenvalues of S_0 (and also of S_0^T) which are of the form: $\lambda = \pm 1/\sqrt{n}$ with $n = 1,\ 2,\ 3,\ \ldots$ due to 2×2-blocks:

$$\begin{pmatrix} 0 & 1/n_1 \\ 1/n_2 & 0 \end{pmatrix} \quad \Rightarrow \quad \lambda = \pm\frac{1}{\sqrt{n_1\, n_2}}\,.$$

2. Those with $d^T\psi \neq 0 \quad \Rightarrow \quad \mathcal{R}(\lambda) = 0$ with the rational function:

$$\boxed{\mathcal{R}(\lambda) = 1 - d^T \frac{\mathbf{1}}{\lambda\,\mathbf{1} - S_0} e/N = 1 - \sum_{j,q} \frac{C_{jq}}{(\lambda - \rho_j)^q}}$$

Here C_{jq} and ρ_j are unknown, except for
$\rho_1 = 2\,\mathrm{Re}\,[(9 + i\sqrt{119})^{1/3}]/(135)^{1/3} \approx 0.9024$ and
$\rho_{2,3} = \pm 1/\sqrt{2} \approx \pm 0.7071$.

Idea: Expand the geometric matrix series $\Rightarrow$

$$\boxed{\mathcal{R}(\lambda) = 1 - \sum_{j=0}^{\infty} c_j \lambda^{-1-j} \quad , \quad c_j = d^T\, S_0^j\, e/N}$$

which converges for $|\lambda| > \rho_1 \approx 0.9024$ since $c_j \sim \rho_1^j$ for $j \to \infty$.

Problem: How to determine the zeros of $\mathcal{R}(\lambda)$ with $|\lambda| < \rho_1$?

Analytic continuation by rational interpolation:

Use the series to evaluate $\mathcal{R}(z)$ at n_S support points $z_j = \exp(2\pi ij/n_S)$ with a given precision of p binary digits and determine the rational function $R_I(z)$ which interpolates $\mathcal{R}(z)$ at these support points. Two cases:

$$n_S = 2n_R + 1 \quad \Rightarrow \quad R_I(z) = \frac{P_{n_R}(z)}{Q_{n_R}(z)}$$

$$n_S = 2n_R + 2 \quad \Rightarrow \quad R_I(z) = \frac{P_{n_R}(z)}{Q_{n_R+1}(z)}$$

The n_R zeros of $P_{n_R}(z)$ are approximations of the eigenvalues of S (of the 2nd group).

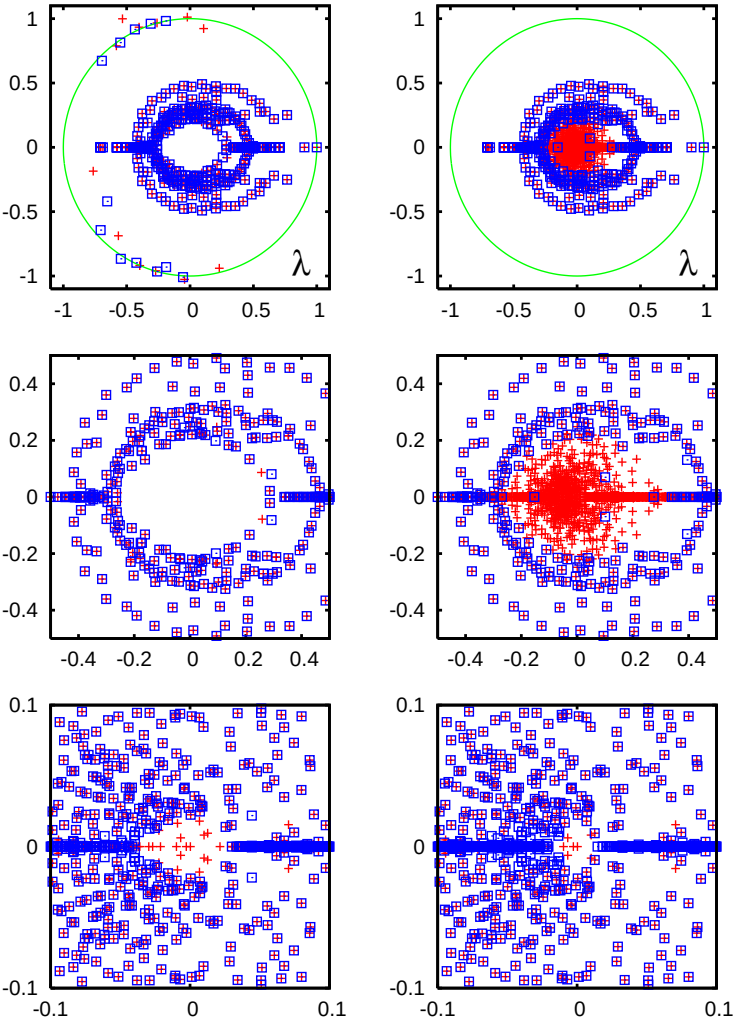
For a given precision, e. g. $p = 1024$ binary digits one can obtain a certain number of reliable eigenvalues, e. g. $n_R = 300$. The method can be pushed up to $p = 16384$ and $n_R = 2500$ which is better than the high precision Arnoldi method with $n_A = 2000$.

Examples:

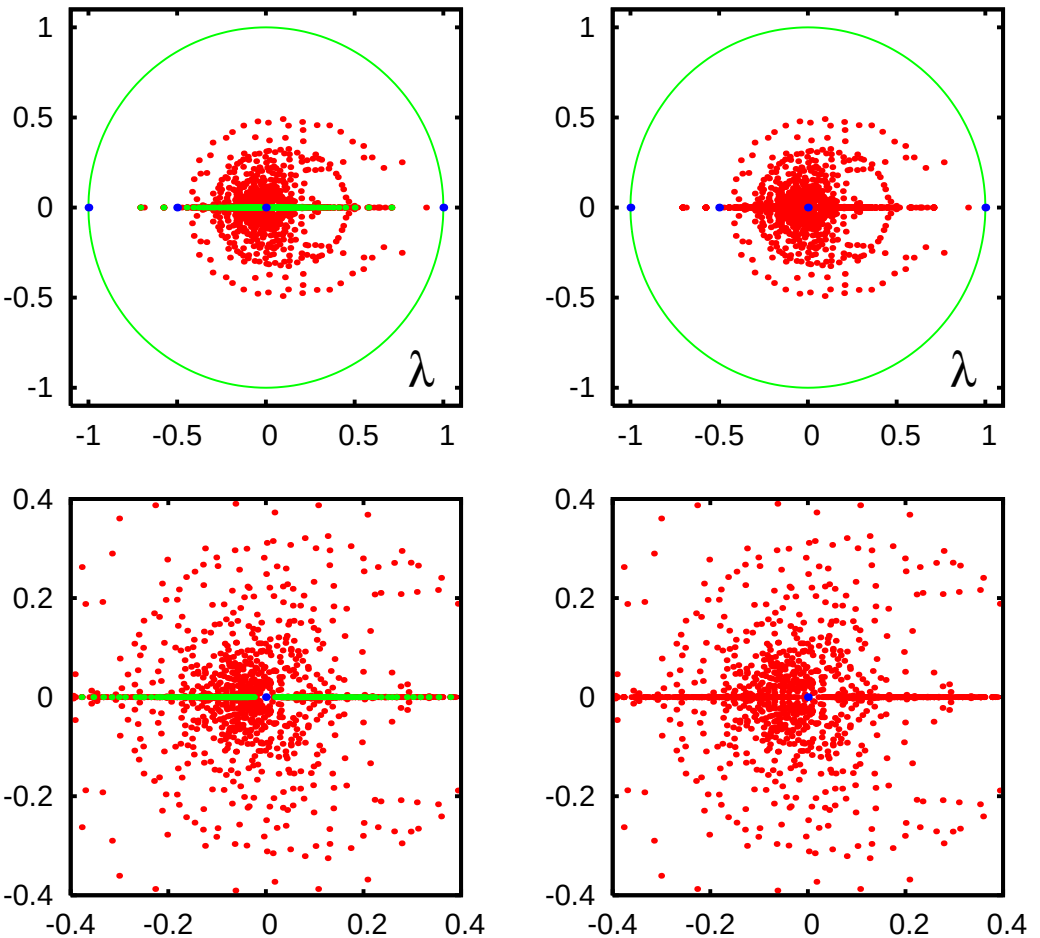
Some “artificial zeros” for $n_R = 340$ and $p = 1024$ (*left top and middle panels*) where both variants of the method differ.

For $n_R = 300$ and $p = 1024$ most zeros coincide with HP Arnoldi method (*right top and middle panels*) and both variants of the method coincide.

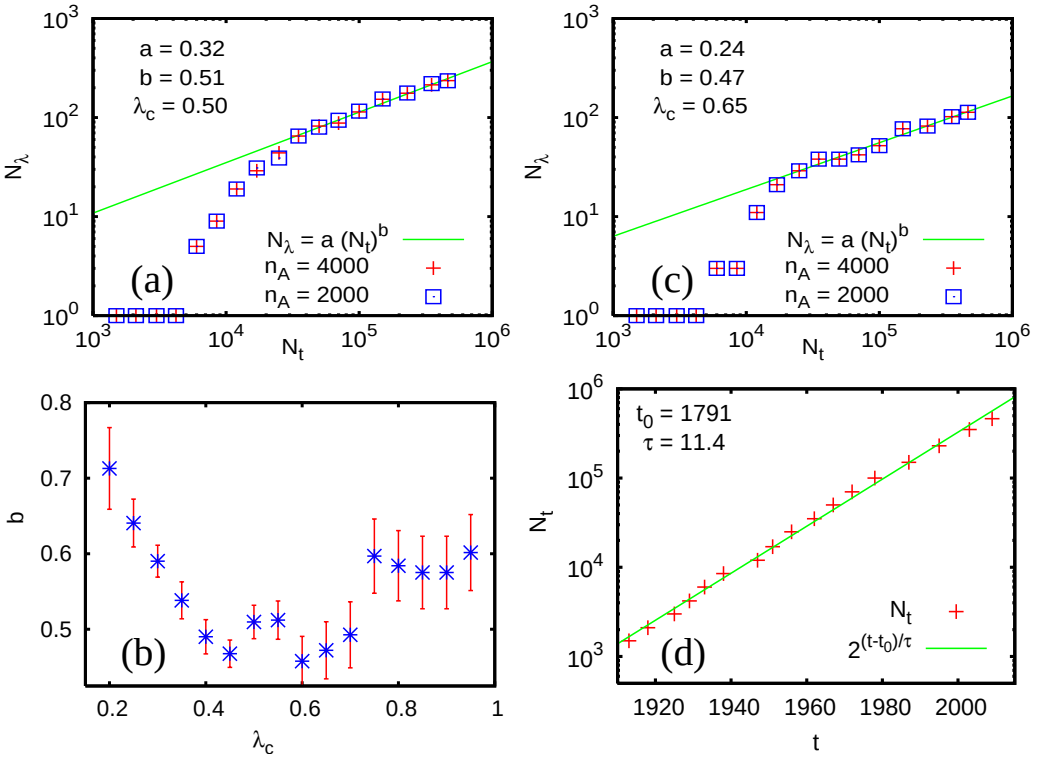
Lower panels: comparison for $n_R = 2000$, $p = 12288$ (left) or for $n_R = 2500$, $p = 16384$ with HP Arnoldi method.



Accurate eigenvalue spectrum for the full Physical Review network by the rational interpolation method (left) and the HP Arnoldi method (right):



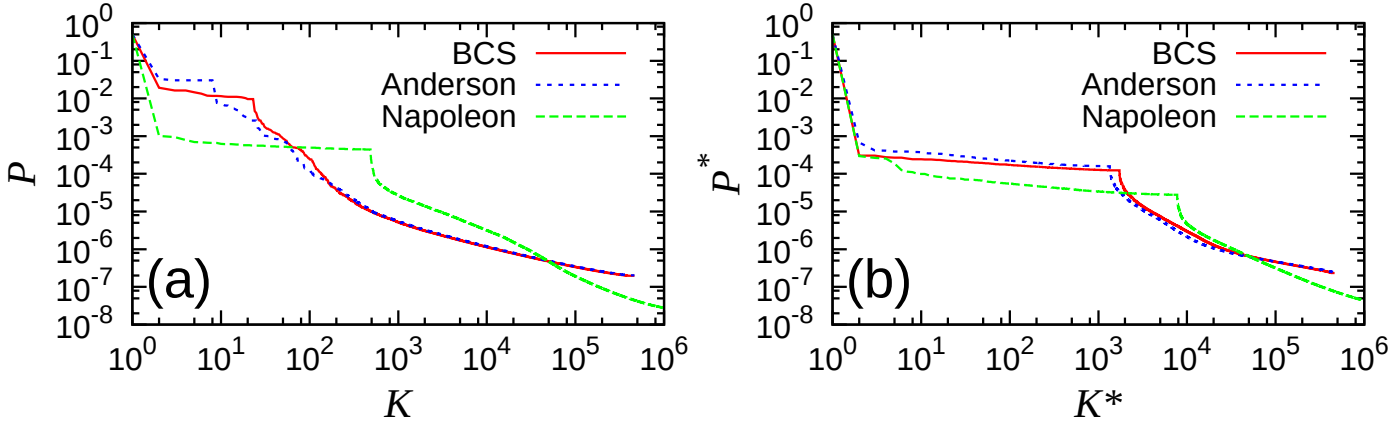
Fractal Weyl law



$N_\lambda =$ number of complex eigenvalues with $\lambda_c \leq |\lambda| \leq 1$.
$N_t =$ reduced network size of Physical Review at time t.

$$N_\lambda = a N_t^b$$

ImpactRank for influence propagation



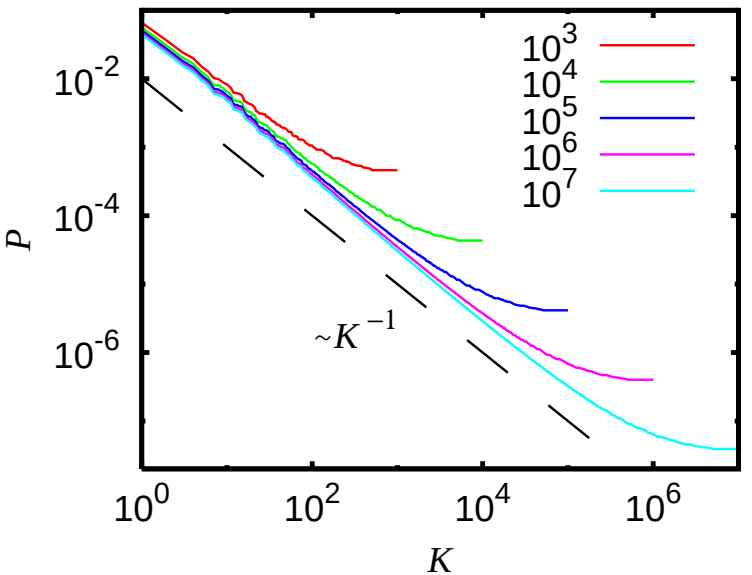
$$v_f = \frac{1-\gamma}{1-\gamma G}\, v_0 \quad , \quad v_f^* = \frac{1-\gamma}{1-\gamma G^*}\, v_0$$

v_f = "PageRank" of $\tilde{G}$ with:

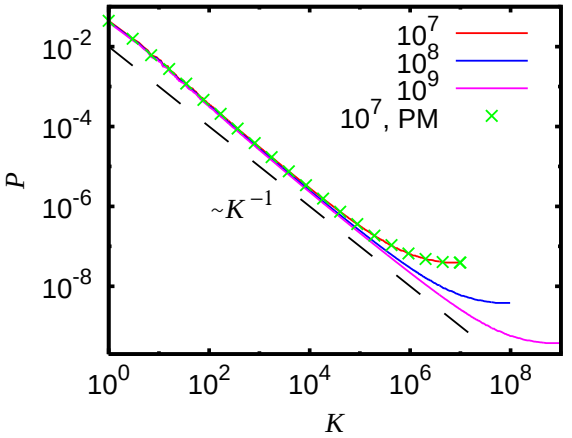
$$\tilde{G} = \gamma\, G + (1-\gamma)\, v_0\, e^T$$

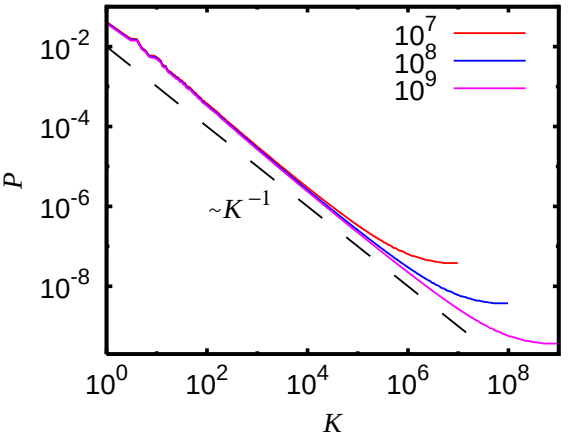
Integer network

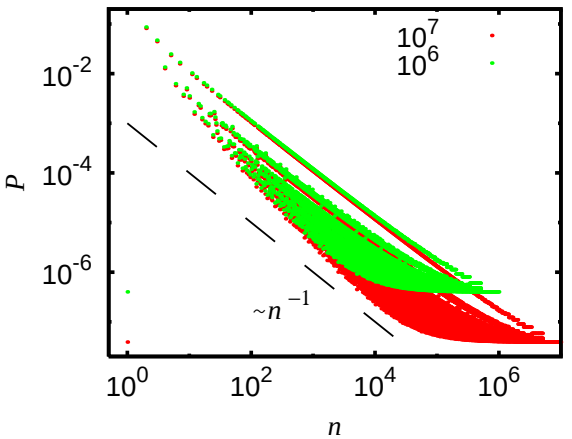
Consider the integers $n \in \{1, \ldots, N\}$ and construct an adjacency matrix by $A_{mn} = k$ where k is the largest integer such that m^k is a divisor of n if $1 < m < n$ and $A_{mn} = 0$ if $m = 1$ or $m = n$ (note $A_{mn} = k = 0$ if m is not a divisor of n). Construct S and G in the usual way from A.

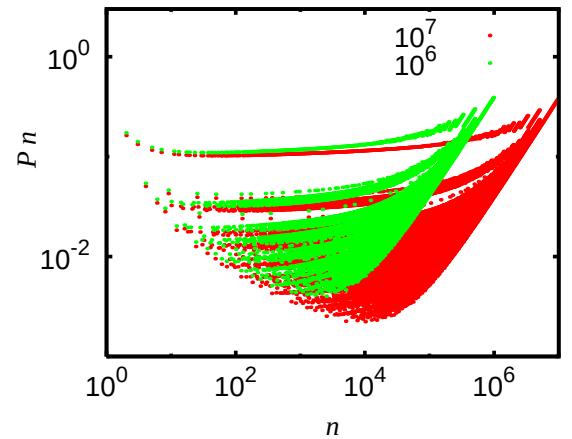


PageRank

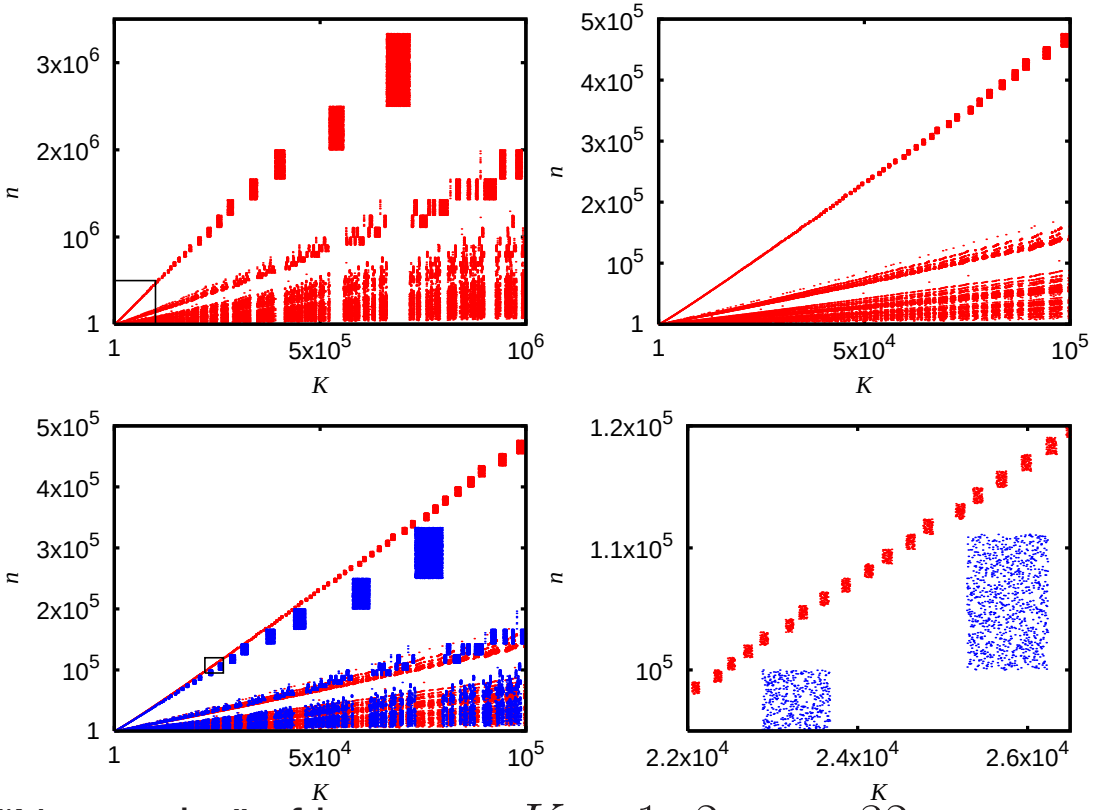

10^7
10^8
10^9
10^7, PM
$\sim K^{-1}$
P
K


10^7
10^8
10^9
$\sim K^{-1}$
P
K


10^7
10^6
$\sim n^{-1}$
P
n


10^7
10^6
$P\,n$
n

Dependence of n on K-index



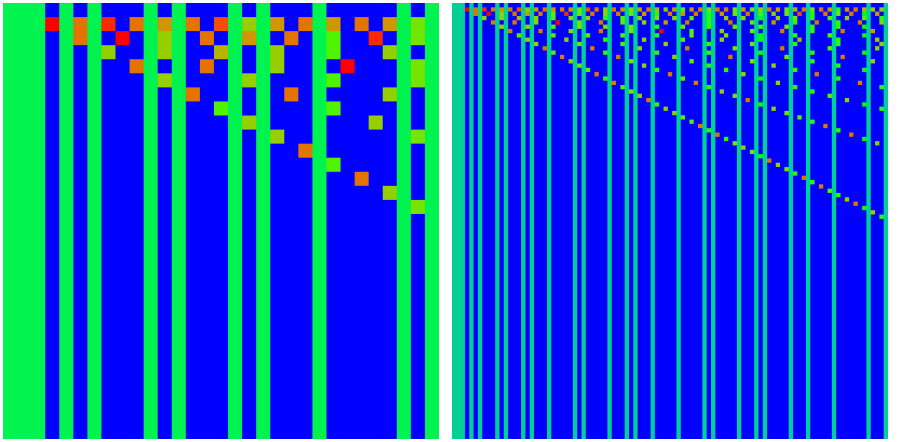
red: $N = 10^7$

blue: $N = 10^6$

"New order" of integers: $K = 1, 2, \ldots, 32 \Rightarrow n = 2, 3, 5, 7, 4, 11, 13, 17, 6, 19, 9, 23, 29, 8, 31, 10, 37, 41, 43, 14, 47, 15, 53, 59, 61, 25, 67, 12, 71, 73, 22, 21.$

Semi-analytical determination of spectrum, PageRank and eigenvectors

Matrix structure:



$$\boxed{S = S_0 + v\, d^T}$$

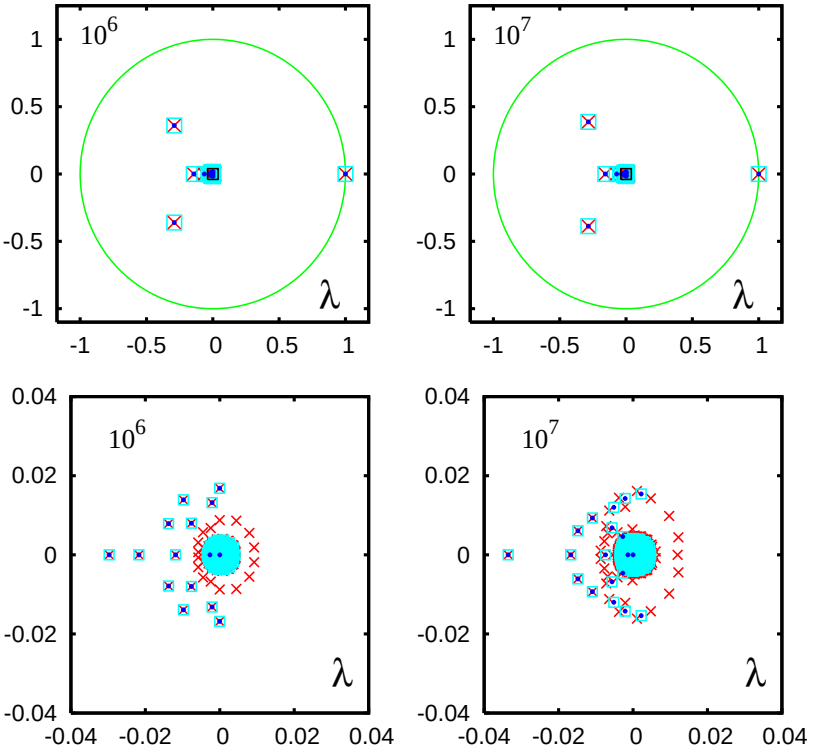
where $v = e/N$, $d_j = 1$ for dangling nodes (primes and 1) and $d_j = 0$ otherwise. S_0 is the pure link matrix which is ***nil-potent***:

$$\boxed{S_0^l = 0}$$

with $l = [\log_2(N)] \ll N$

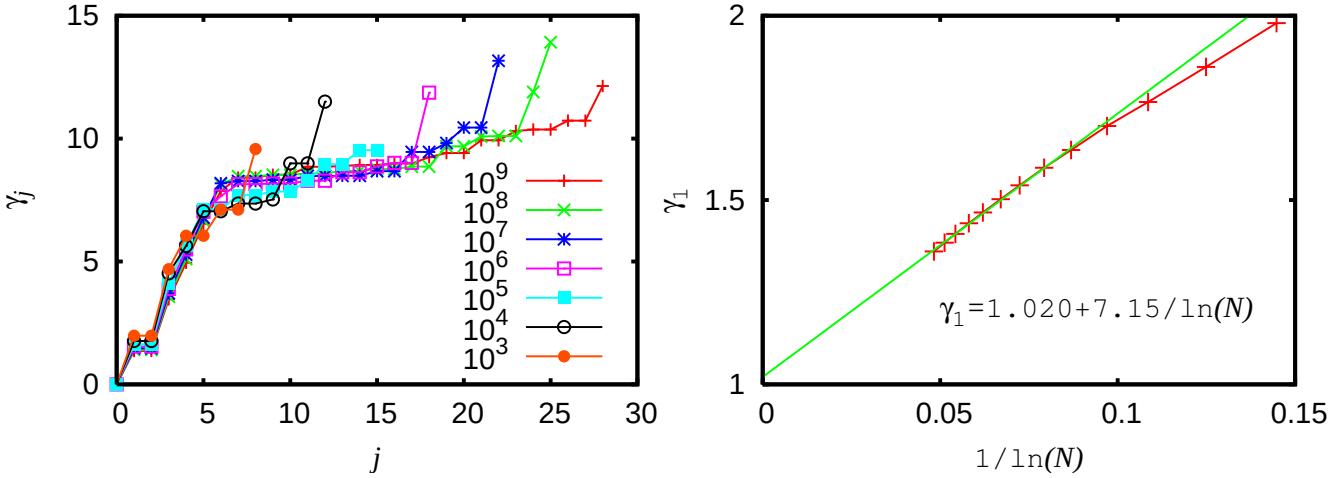
$\Rightarrow$ same theory as for the Phys.-Rev. Network.

Spectrum I



blue dots: semi-analytical eigenvalues as zeros from $\mathcal{P}_r(\lambda)$ (or eigenvalues of $\bar{S}$).
red crosses: Arnoldi method with random initial vector and $n_A = 1000$.
light blue boxes: Arnoldi method with constant initial vector $v = e/N$ and $n_A = 1000$.

Spectrum II



$\gamma_j = -2 \ln |\lambda_j|$

Large N limit of γ_1 with the scaling parameter: $1/\ln(N)$.

Note:

$$c_0 = d^T v = \frac{1}{N} \sum_{j=1}^{N} d_j = \frac{1 + \pi(N)}{N} \approx \frac{1}{\ln(N)}$$

where $\pi(N)$ is the number of primes below N.

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3. K. M. Frahm, Y. H. Eom, and D. L. Shepelyansky, **Google matrix of the citation network of Physical Review**, Phys. Rev. E **89**, 052814 (2014).