UNIVERSITY OF TWENTE.

Quick detection of popular entities in large on-line networks

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- Facebook has more than 1 billion users. With an average user having 190 friends, the number of social links in Facebook is 190 billion.

- The static part of the web graph has more than 10 billion pages. With an average number of 38 hyper-links per page, the total number of hyper-links is 380 billion.

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  - It is simply interesting!

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If the adjacency list of the network is known...

the top-k list of nodes can be found by the HeapSort with complexity $O(N + klog(N))$, where N is the total number of nodes.

Even this modest complexity can be demanding for large networks.

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Avrachenkov, L, Sokol, Towsley (2012); Cooper, Radzik, Siantos (2012), Borgs, Brautbar, Chayes, Khanna, Lucier (2012),
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Goal: Find top-k most popular entities in social (directed) networks (nodes with highest in/out-degrees, largest interest groups, largest user categories), using the minimal number of API requests.

Problem formulation

- Consider a bi-partite graph (V, W, E)
- V and W are sets of entities, $|V| = M$, $|W| = N$.
- A directed edge $(v, w) \in E$ represents a relation between $v \in V$ and $w \in W$.
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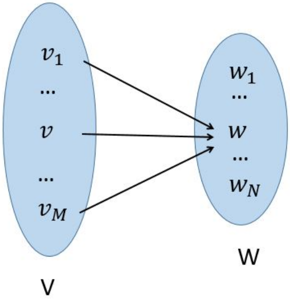

Example. $V = W$ is a set of Twitter users, (v, w) means that v follows w.

Example. V is a set of users, W is a set of interest groups, (v, w) means that user v is a member of an interest group w.

Algorithm for finding top-k most popular entities

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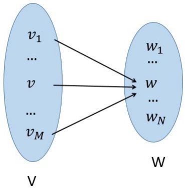
1. Choose a set $A \subset V$ of n_1 nodes sampled from V at random.
2. For each $v \in A$ retrieve the id's of nodes in W that have an edge from v.
3. Compute S_w – the number of edges of $w \in W$ from A.
4. Retrieve the actual degrees for the n_2 nodes w with the largest values of S_w.
5. Return the identified top-k list of most popular entities in W.



In total, we use $n = n_1 + n_2$ requests to API (Step 2 and Step 4).

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- In a randomly chosen set of n_1 Twitter users only a few users follow more than 5000 people. Thus, we retrieve at most 5000 followees of each node. This does not affect the results.
- Make a guess: We use 1000 requests to API. For which k can we identify a top-k list of most followed Twitter users with 90% precision?

Results

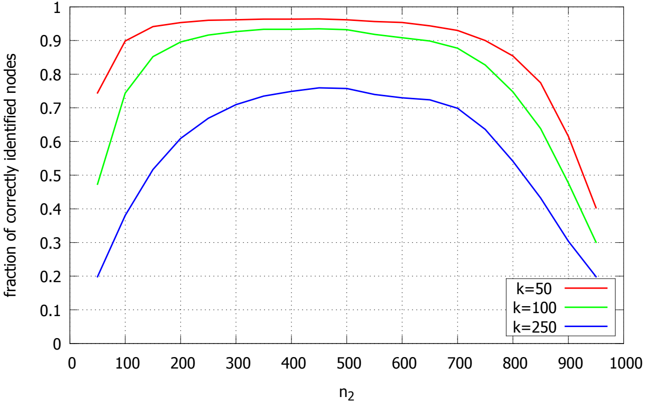

Figure : The fraction of correctly identified top-k most followed Twitter users as a function of n_2, with $n = 1000$.

Most followed

Twitter users		Followers	Following	Tweets
1	KATY PERRY @katyperry	53,923,965	148	5,699
2	Justin Bieber @justinbieber	52,445,383	130,204	27,064
3	Barack Obama @BarackObama	43,712,727	650,033	11,955
4	YouTube @YouTube	43,007,224	704	10,599
5	Lady Gaga @ladygaga	41,548,506	134,424	4,782

Interest groups VKontakte

- Popular social network in Russian, more than 200M users.

Rank	Number of participants	Topic
1	4,35M	humor
2	4,1M	humor
3	3,76M	movies
4	3,69M	humor
5	3,59M	humor
6	3,58M	facts
7	3,36M	cookery
8	3,31M	humor
9	3,14M	humor
10	3,14M	movies
100	1,65M	success

- With $n_1 = 700$, $n_2 = 300$, our algorithm identifies on average 73.2 from the top-100 interest groups (averaged over 25 experiments). The standard deviation is 4.6.

Comparison to known algorithms

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- How our algorithm compares to baselines?

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- Transitions probabilities along undirected edges (x, y) are proportional to $(d(x)d(y))^b$, where $d(x)$ is the degree of a vertex x and $b > 0$ is some parameter.

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- Not implementable on Twitter

Random Walk

Avrachenkov, L, Sokol, Towsley (2012)

- Random walk with uniform jumps:

$$p(x,y) = \begin{cases} \frac{\alpha/N+1}{d(x)+\alpha}, & \text{if } x \text{ has a link to } y, \\ \frac{\alpha/N}{d(x)+\alpha}, & \text{if } x \text{ does not have a link to } y, \end{cases}$$

  where N is the number of nodes in the graph and $d(x)$ is the degree of a node x.
- Rationale: in undirected graphs the stationary distribution is given by

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- Best to take α approximately equal to the average degree

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- Strict: [one step of the algorithm] = [one API request]
- Relaxed: [one step of the algorithm] = [one considered vertex]

Crawl-AI and Crawl-GAI

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- Can suffer from correlations between in- and out-degrees

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- A lot of resources are spent on out-neighbors of random nodes

Comparison of the algorithms

Table : Percentage of correctly identified nodes from top-100 in Twitter averaged over 30 experiments, $n = 1000$

Algorithm	mean	standard deviation
Two-stage algorithm	92.6	4.7
Random walk (strict)	0.43	0.63
Random walk (relaxed)	8.7	2.4
Crawl-GAI	4.1	5.9
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Advantages of the two-stage algorithm:

- does not waste resources
- obtains *exact* degrees of the n_2 'most promising' nodes

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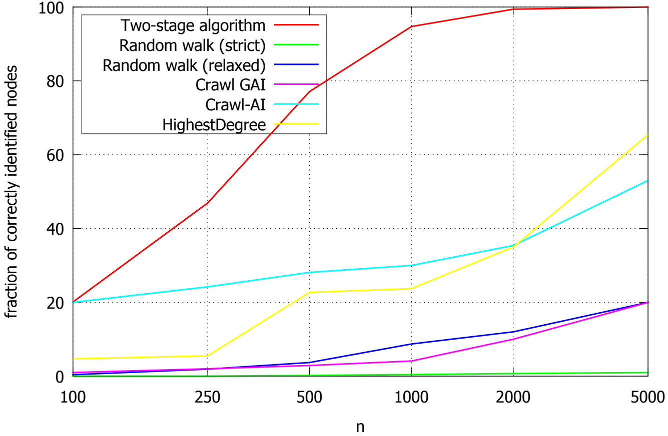

Figure : The fraction of correctly identified top-100 most followed Twitter users as a function of *n* averaged over 10 experiments.

Influence of graph size?

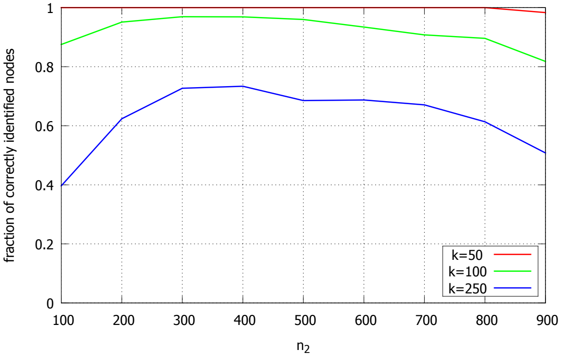

Figure : The fraction of correctly identified top-k in-degree nodes in the CNR-2000 graph (*law.di.unimi.it/webdata/cnr-2000*) as a function of n_2, with $n = 1000$. Note that algorithm performs similarly on CNR-2000 (half a million nodes) and Twitter.

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Formal view on the hubs

Let D be a degree of a random node. Regular varying distribution:

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$$\left(\frac{F_1}{a_N}, \cdots, \frac{F_k}{a_N}\right) \xrightarrow{d} \left(\frac{E_1^{-\delta}}{\delta}, \cdots, \frac{\left(\sum_{i=1}^{k} E_i\right)^{-\delta}}{\delta}\right),$$

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- Performance measure:

$$E[\text{fraction of correctly identified top-}k\text{ entities}]$$

$$= \frac{1}{k} \sum_{j=1}^{k} P(j \in \{i_1, \ldots, i_{n_2}\}). \quad (2)$$

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$$= \frac{1}{k} \sum_{j=1}^{k} P(j \in \{i_1, \ldots, i_{n_2}\}). \qquad (2)$$

- Computation of $P(j \in \{i_1, \ldots, i_{n_2}\})$ is not feasible even if degrees are known

Poisson prediction

- $P(j \in \{i_1, \ldots, i_{n_2}\})$
  $= P(S_j > S_{i_{n_2}}) + P(S_j = S_{i_{n_2}}, j \in \{i_1, \ldots, i_{n_2}\})$
- **Example.** Twitter graph, take $n_1 = n_2 = 500$. Then the average number of nodes i with $S_i = 1$ among the top-l nodes is

$$\sum_{i=1}^{l} P(S_i = 1) = \sum_{i=1}^{l} 500 \, \frac{F_i}{5 \cdot 10^8} \left(1 - \frac{F_i}{5 \cdot 10^8}\right)^{499},$$

  which is 2540.6 for $l = 10,000$ and it is 57.4 for $l = n_2 = 500$. Hence, typically, $[S_{i_{500}} = 1]$. The event $[i \in \{i_1, \ldots, i_{n_2}\}]$ occurs only for a small fraction of nodes i with $[S_i = 1]$.

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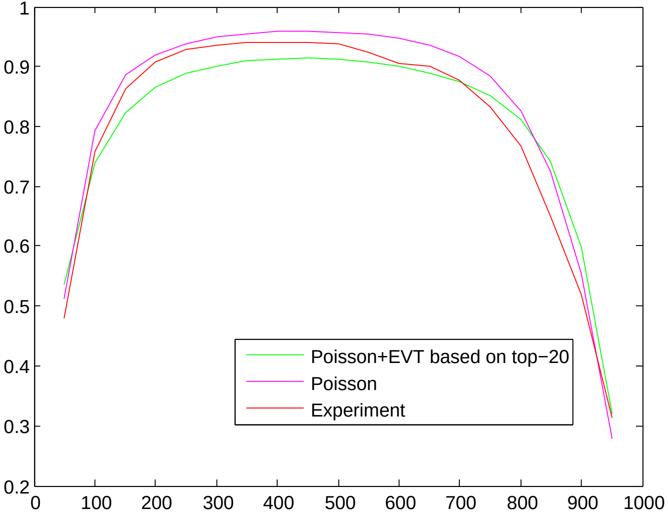
- Estimator for high degrees: Dekkers et al. (1989)
  $\hat{f}_j = \hat{F}_m \left(\frac{m}{j-1} \right)^{1/\hat{\gamma}}, \qquad j > 1, j << N.$
- Use $S_j \sim Poisson(n_1 \hat{f}_j / N)$

Performance predictions on the Twitter graph



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- High variability helps a lot!

Thank you!