On Computing the Diameter of Real World Networks

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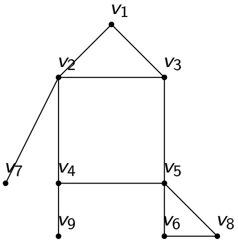
Given a graph $G = (V, E)$ connected.

- The distance $d(u, v)$ is the number (sum of the weights) of edges along the shortest path from u to v.
- The eccentricity of a node u, $\mathrm{ecc}(u) = \max_{v \in V} d(u, v)$: in how many hops u can reach any node?

Definition (Diameter)

$$D = \max_{u,v \in V} d(u, v) = \max_{u \in V} \mathrm{ecc}(u)$$



	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	ecc
v_1	0	1	1	2	2	3	2	3	3	3
v_2	1	0	1	1	2	3	1	3	2	3
v_3	1	1	0	2	1	2	2	2	3	3
v_4	2	1	2	0	1	2	2	2	1	2
v_5	2	2	1	1	0	1	3	1	2	3
v_6	3	3	2	2	1	0	4	1	3	4
v_7	2	1	2	2	3	4	0	4	3	4
v_8	3	3	2	2	1	1	4	0	3	4
v_9	3	2	3	1	2	3	3	3	0	3

Why people compute the diameter?

- It gives bounds on information diffusion time, e.g., lower bound on how quickly information reaches **EVERY** individual.

How they do it?

- Textbook Algorithm ($n = |V|$, $m = |E|$). Too expensive.
    - Perform a Breadth First Search (BFS) for each node and return maximum ecc.
        - A BFS from x returns all the distances from x and takes $O(m)$ time.
- All other approaches (see [Zwick, 2001]) solves also all pairs shortest path: they have to require at least $\Omega(n^2)$. Still too expensive.
- Otherwise approximation algorithms or heuristics, as for instance empirically finding lower bound L and upper bound U.
    - That is, $L \leq D \leq U$
    - D found, when $L = U$

Bound Refinement: iterative fringe upper bound

Recall that

The *textbook* algorithm runs a BFS for any node and return the maximum ecc found.

iFUB is a particular case in which we:

- specify the order in which the BFSes have to be executed
- refine a lower bound,
  - that is the maximum ecc found until that moment.
- upper bound the eccentricities of the remaining nodes.
- stop when the remaining nodes cannot have eccentricity higher than our lower bound.

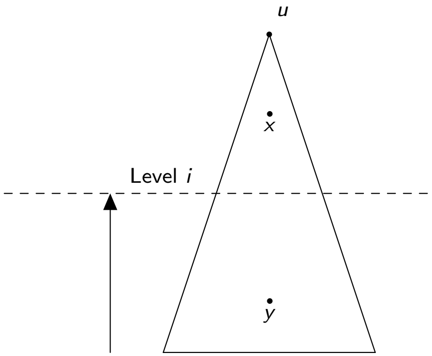
Good order by analyzing how nodes are placed in the BFS tree of a central node u.

- u is the highest degree node or the node returned by some heuristic (2-Sweep or 4-Sweep).

Theorem

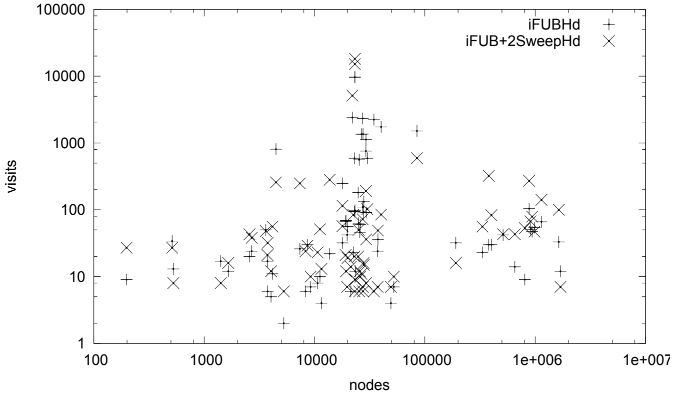
For any $1 \leq i < \mathrm{ecc}(u)$ and $1 \leq k < i$, and for any $x \in N_{i-k}(u)$ such that $\mathrm{ecc}(x) > 2(i-1)$, there exists $y \in N_j(u)$ such that $d(x,y) = \mathrm{ecc}(x)$ with $j \geq i$.

All the nodes x above the level i in $\mathrm{BFS}(u)$ having ecc greater than $2(i-1)$ have a corresponding node y, whose ecc is greater or equal to $\mathrm{ecc}(x)$, below or on the level i in $\mathrm{BFS}(u)$.



Experiments for undirected graphs

The number of visits seems to be constant.

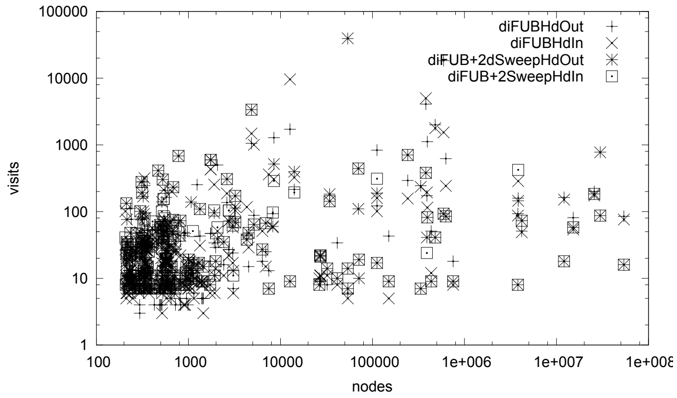


It has been used to compute the diameter of Facebook (721.1M nodes and 68.7G edges, Diameter 41) with just 17 BFSes.

Experiments for directed graphs

The algorithm has been extended also to directed graphs (SEA 2012): in this case it is called directed *i*FUB.

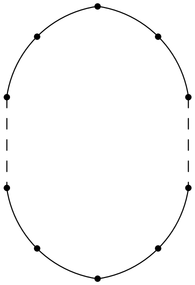
The number of visits seems to be constant.



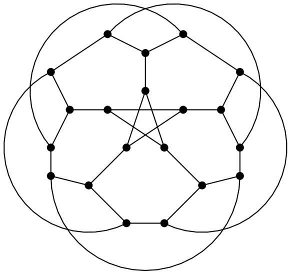
Bad cases

Cases in which nodes have close eccentricity or the BFS trees are isomorphic.

A cycle



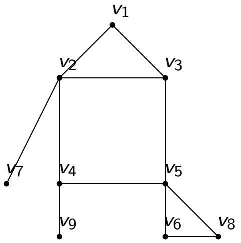
Special regular graphs [such as Moore graphs]



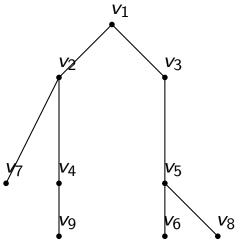
Thanks

Computing lower bound and upper bound easily

A graph



A BFS tree from v_1



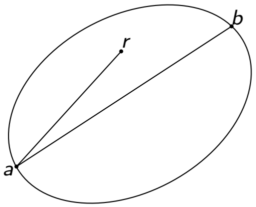
Lower bound	The eccentricity, ecc (height of the BFS tree) of a node. *In the example 3: at least a pair is at distance 3.*
Upper bound	The double of the eccentricity ecc of a node. *In the example 6: every node can reach another node going to v_1 by ≤ 3 edges and going to the destination in ≤ 3 edges.*

- Bounds by sampling are often not tight: $L < D < U$ (see SNAP experiments)
  In the example diameter is 4: $d(v_7, v_8) = 4$.

Good lower bounds in undirected graphs: 2-Sweep

2-Sweep

1. Run a BFS from a random node r: let a be the farthest node.
2. Run a BFS from a: let b be the farthest node.
3. Return the length of the path from a to b.

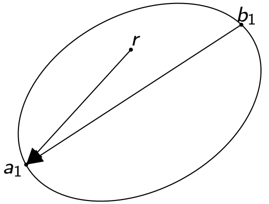


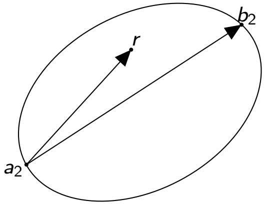
Return $d(a, b)$.

Good lower bounds in directed graphs: directed 2-Sweep

2-*d*Sweep

1. Run a forward BFS from a random node r: let a_1 be the farthest node.
2. Run a backward BFS from a_1: let b_1 be the farthest node.
3. Run a backward BFS from r: let a_2 be the farthest node.
4. Run a forward BFS from a_2: let b_2 be the farthest node.
5. If $\mathrm{ecc}_B(a_1) > \mathrm{ecc}_F(a_2)$, then return the length of the path from b_1 to a_1. Otherwise return the length of the path from a_2 to b_2.

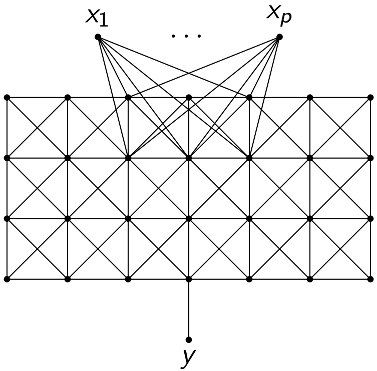




Return the maximum between $d(a_2, b_2)$ and $d(b_1, a_1)$.

First time used in directed graph by Broder et al. to study *Graph structure in the web*.

Bad cases for 2-Sweep



In this modified grid with k rows and $1 + 3k/2$ columns. The algorithm returns $k + 1$. The diameter of the network is instead $3k/2$.