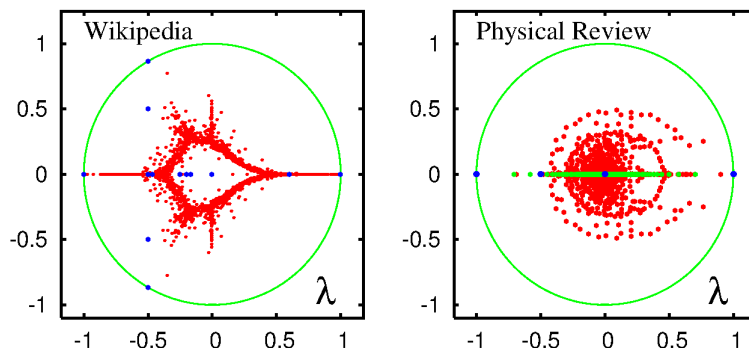




UNIVERSITÉ
TOULOUSE III
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Google matrix analysis of directed networks

Lecture 2

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Networks and data mining

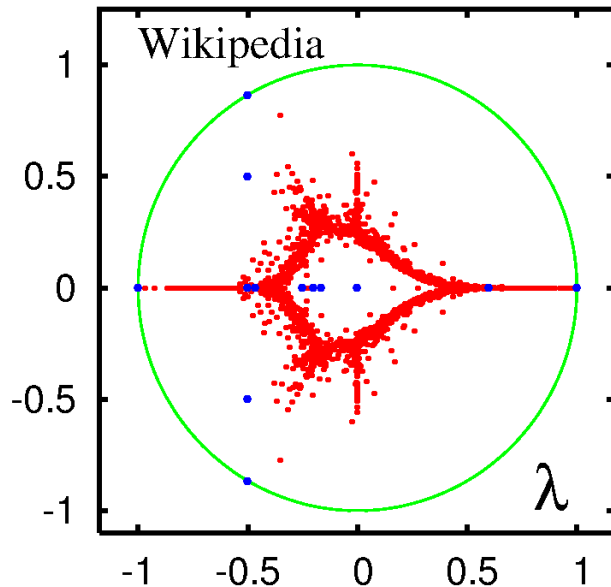
Luchon, June 27 - July 11, 2015

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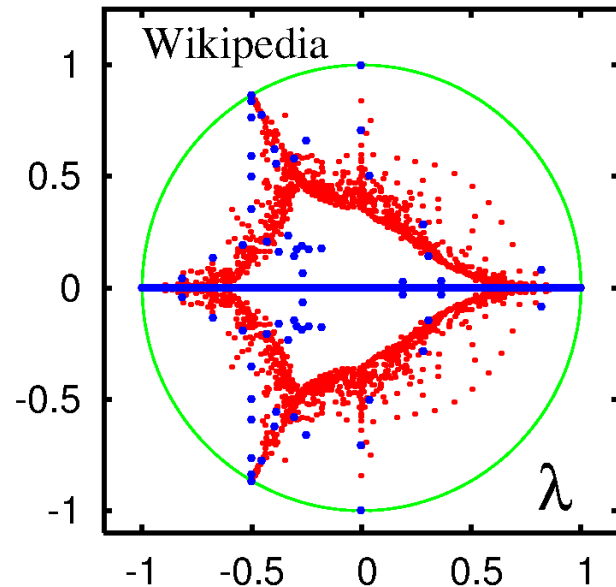
Spectrum of Wikipedia

Wikipedia 2009 : $N = 3282257$ nodes, $N_\ell = 71012307$ network links.



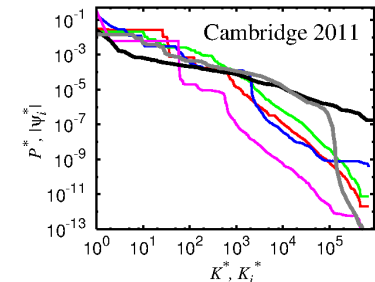
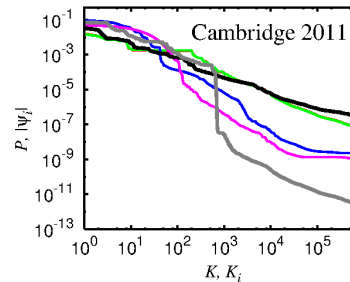
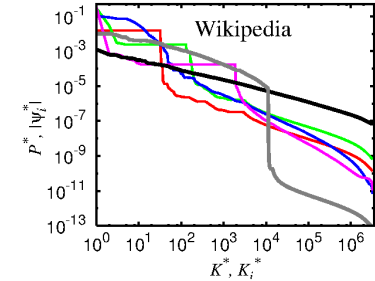
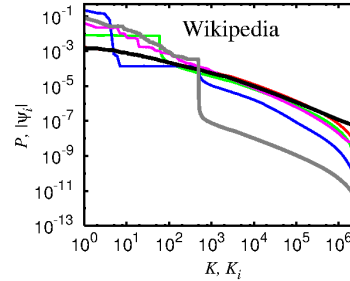
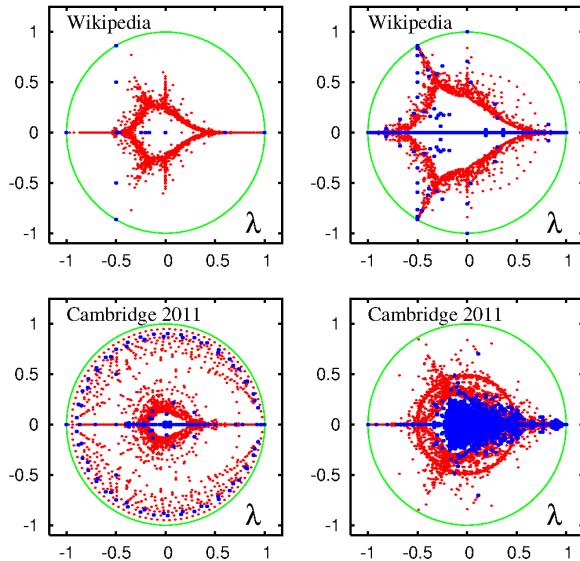
spectrum of S , $N_s = 515$

$n_A = 6000$ for both cases



spectrum of S^* , $N_s = 21198$

Eigenvectors of Wikipedia



left (right): PageRank (CheiRank)

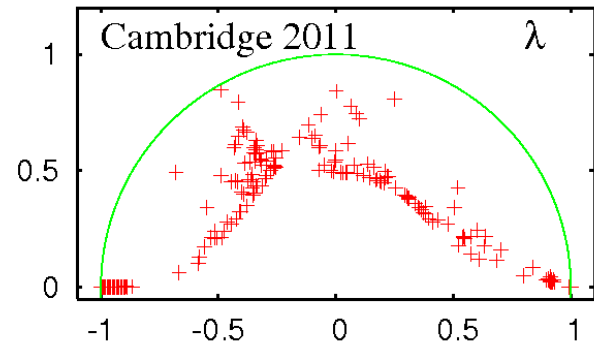
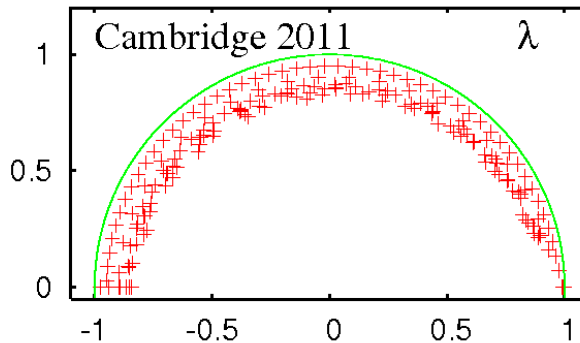
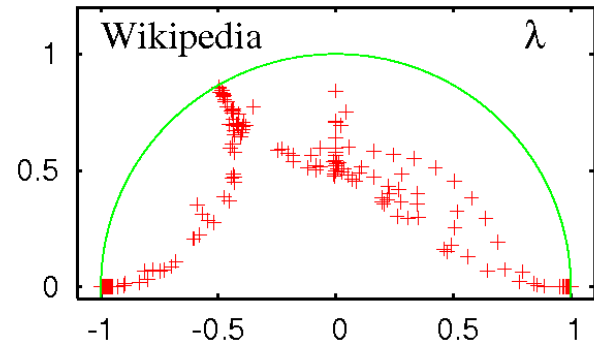
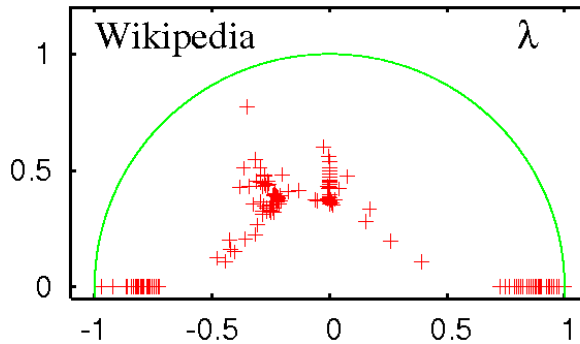
black: PageRank (CheiRank) at $\alpha = 0.85$

grey: PageRank (CheiRank) at $\alpha = 1 - 10^{-8}$

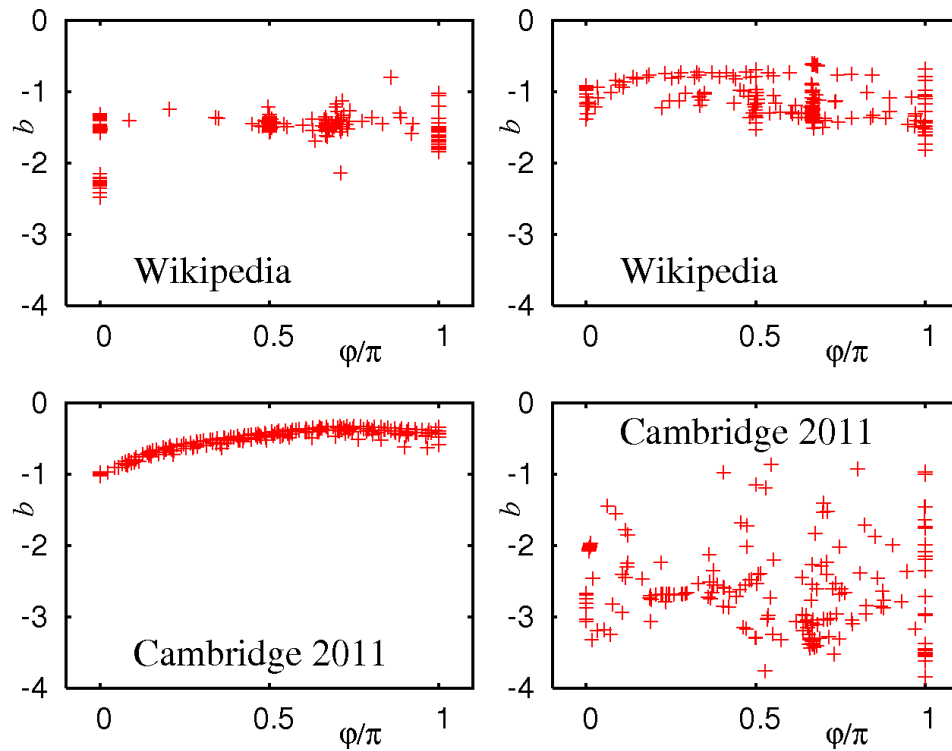
red and green: first two core space eigenvectors

blue and pink: two eigenvectors with large imaginary part in the eigenvalue

Detail study of 200 selected eigenvectors
with eigenvalues “close” to the unit circle:



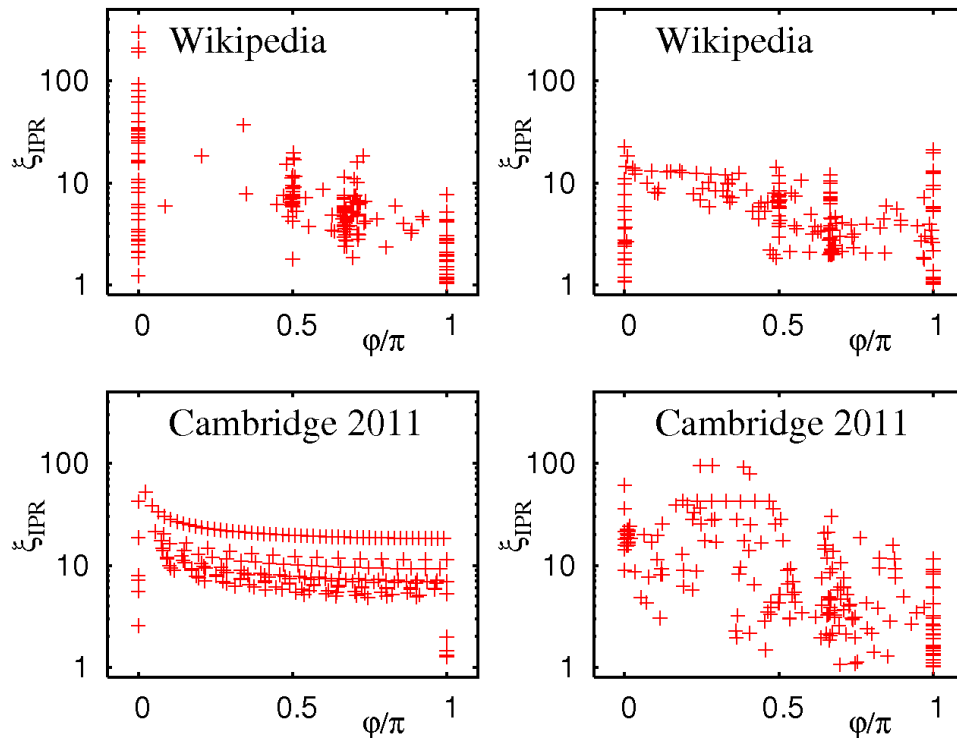
Power law decay of eigenvectors:



$$|\psi_i(K_i)| \sim K_i^b \quad \text{for} \quad K_i \geq 10^4$$

$$\varphi = \arg(\lambda_i)$$

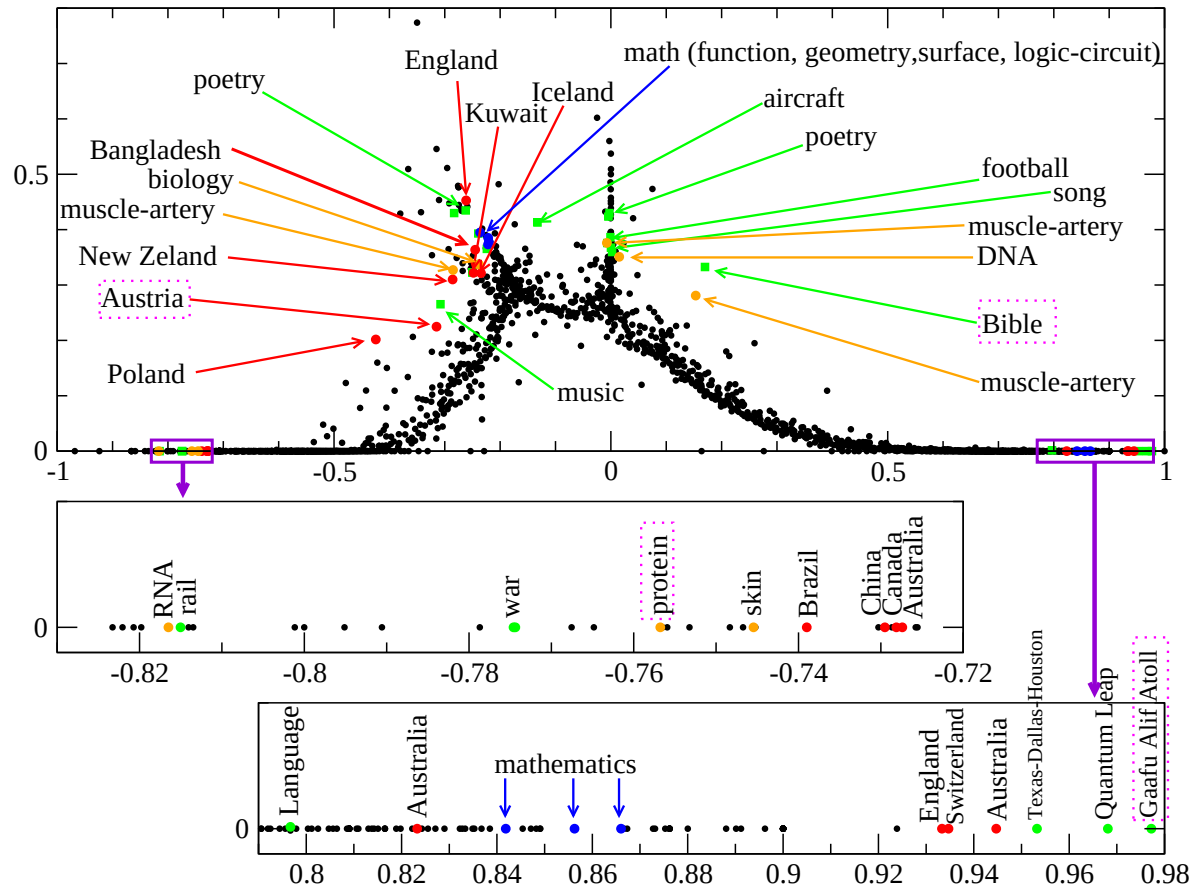
Inverse participation ratio of eigenvectors:



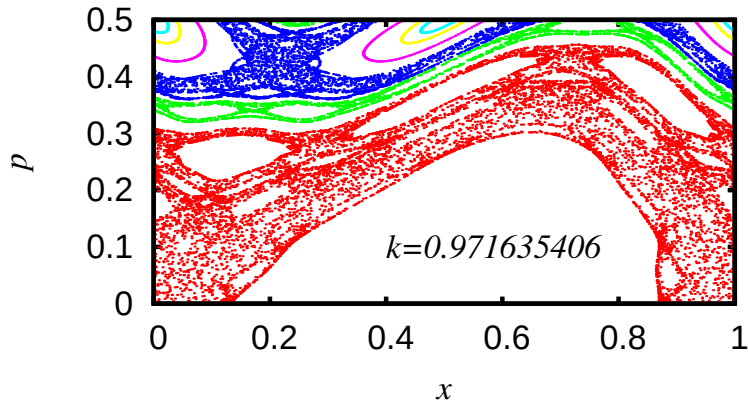
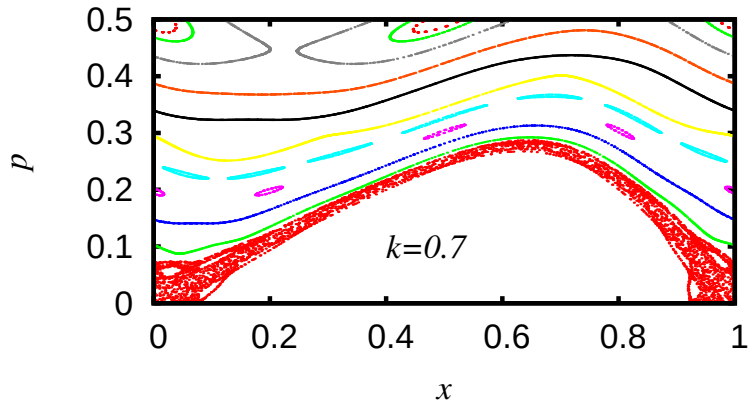
$$\xi_{\text{IPR}} = \left(\sum_j |\psi_i(j)|^2 \right)^2 / \sum_j |\psi_i(j)|^4$$

$$\varphi = \arg(\lambda_i)$$

“Themes” of certain eigenvectors:



Chirikov Standard map



$$p_{n+1} = p_n + \frac{k}{2\pi} \sin(2\pi x_n)$$

$$x_{n+1} = x_n + p_{n+1}$$

x and p are taken modulo 1
and the symmetry

$$(x, p) \rightarrow (1 - x, 1 - p)$$

allows to restrict:

$$x \in [0, 1] \text{ and } p \in [0, 0.5].$$

Transition to “global” chaos at
 $k_c = 0.971635406$.

Perron-Frobenius matrix for chaotic maps

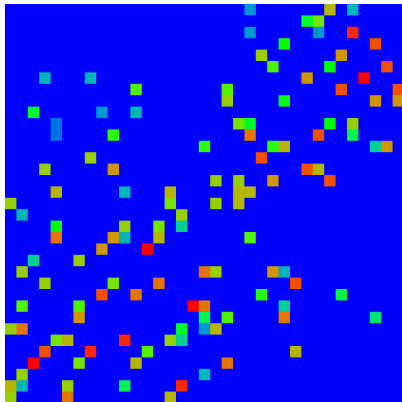
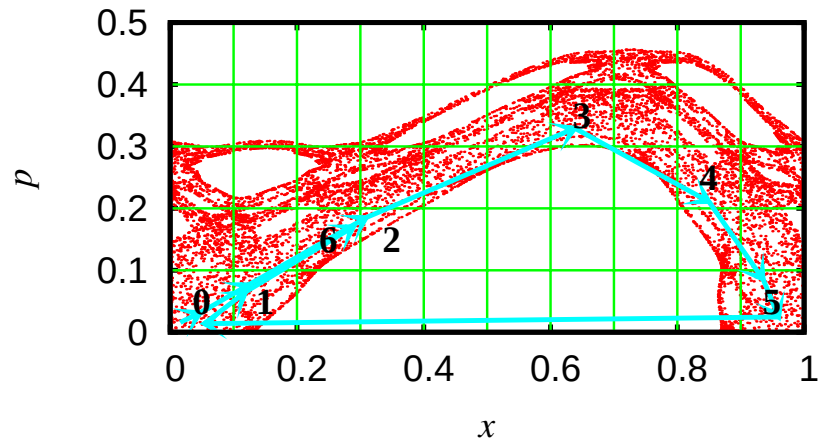
A new variant of the *Ulam Method* to construct the *Perron-Frobenius matrix* for the case of a mixed phase space:

- Subdivide x space in M cells and p space in $M/2$ cells with M being an (even) integer number.
- Iterate (for a very long time: $t \sim 10^{11} - 10^{12}$) a classical trajectory and attribute a new number to each new cell which is entered. At the same time count the number of transitions from cell i to cell j ($\Rightarrow n_{ji}$).
- Calculate the $N \times N$ matrix

$$G_{ji} = \frac{n_{ji}}{\sum_l n_{li}}$$

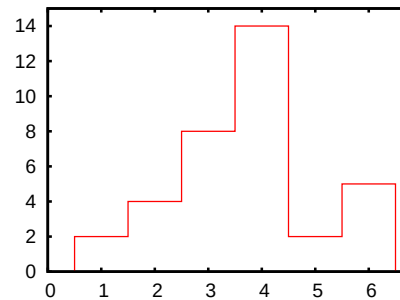
of dimension $N \approx M^2/4$ and which is a (sparse) Perron Frobenius operator, i. e.: $G_{ji} \geq 0$, $\sum_j G_{ji} = 1$, G_{ji} sparse.

$$M = 10, t = 10^6 \Rightarrow N = 35$$



density plot of matrix elements

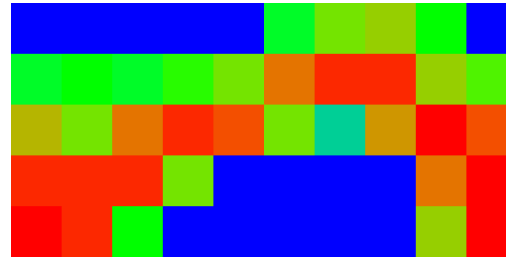
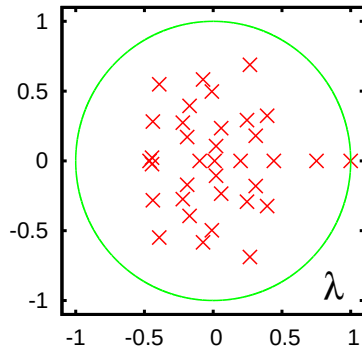
(blue=min=0, green=medium, red=max)



distribution of number of non-zero matrix elements per column

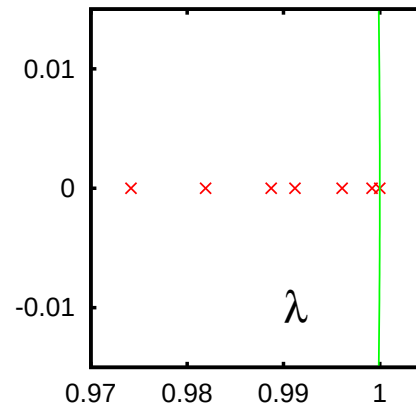
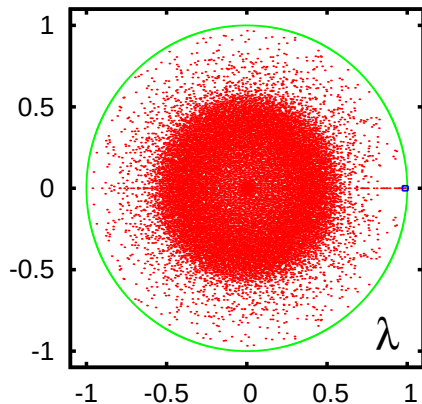
Eigenvalues

for $M = 10$, $t = 10^6$ and $N = 35$

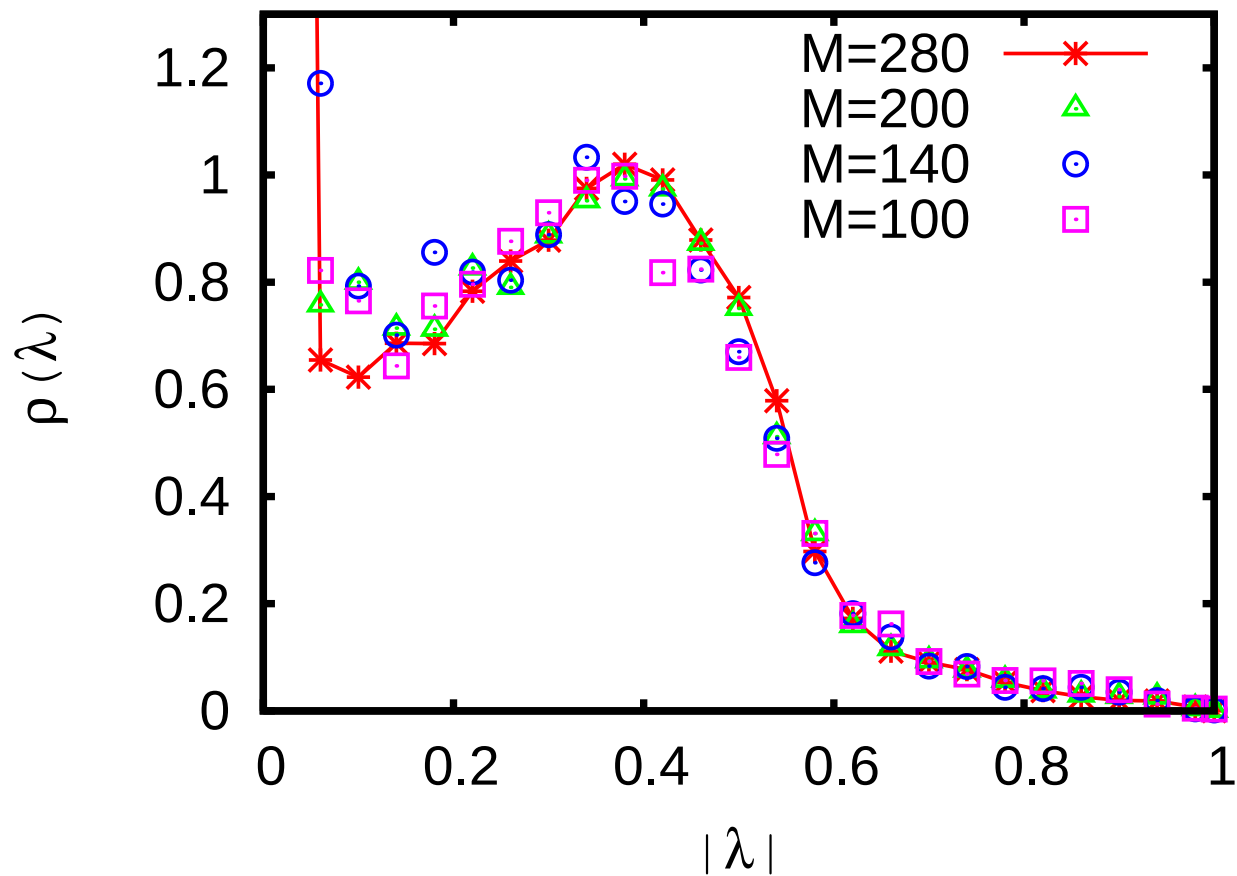


Phase space representation of the eigenvector for $\lambda_0 = 1$.

for $M = 280$, $t = 10^{12}$ and $N = 16609$

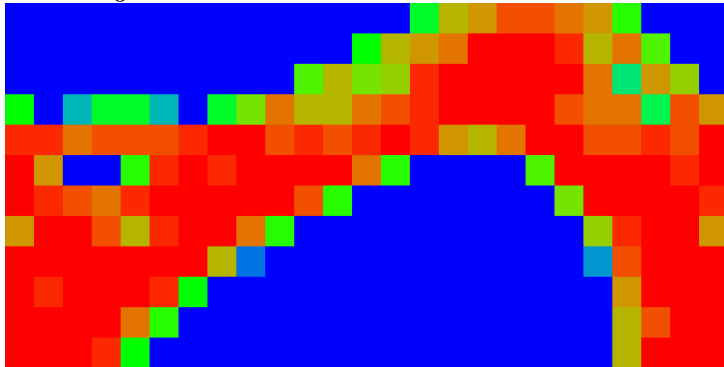


Complex density of states

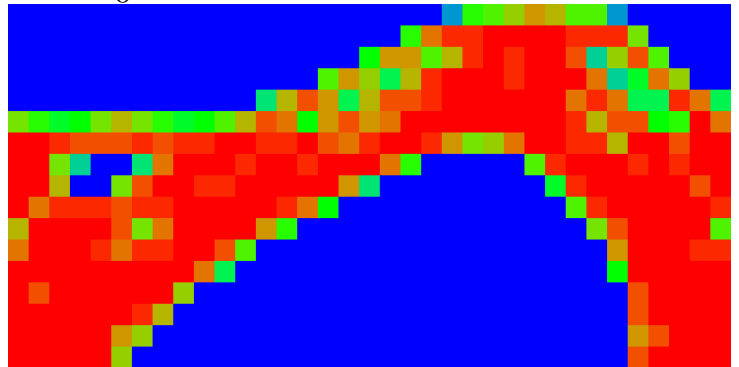


Eigenvectors

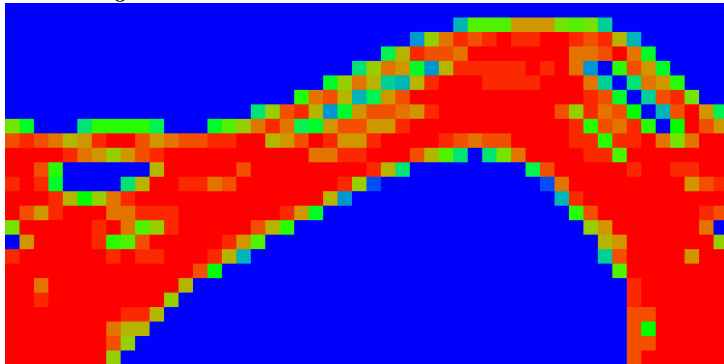
$\lambda_0 = 1, M = 25, N = 177$



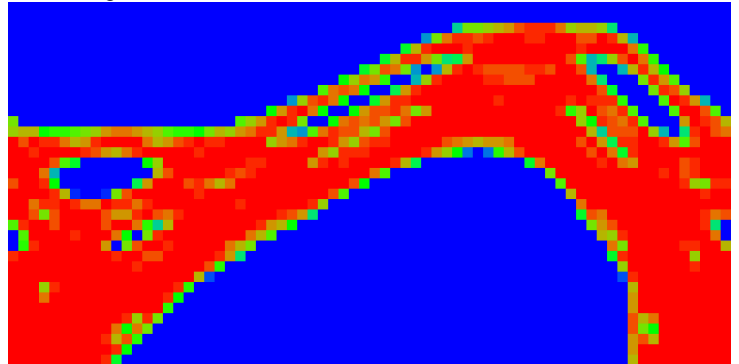
$\lambda_0 = 1, M = 35, N = 332$

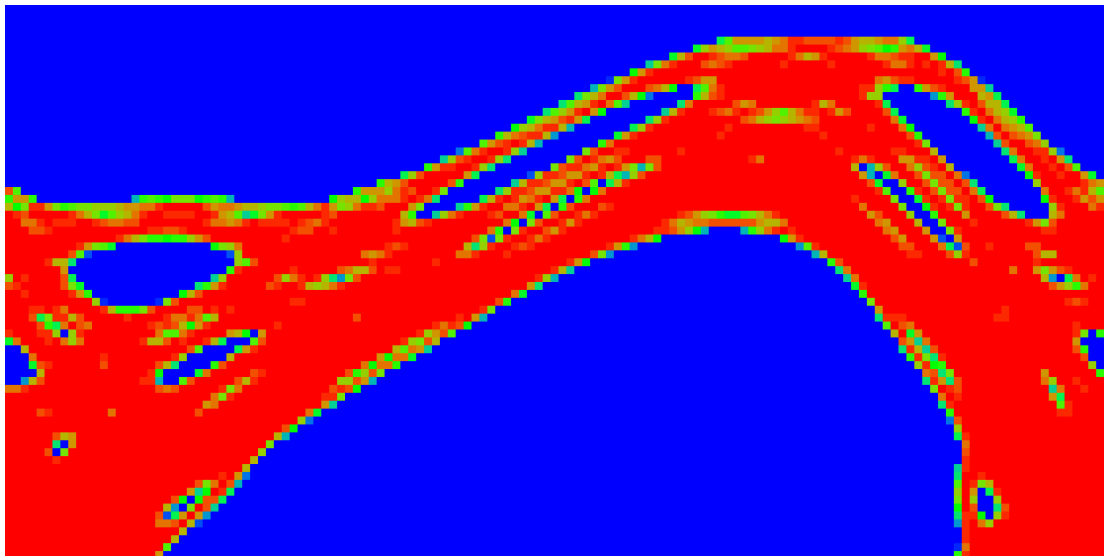


$\lambda_0 = 1, M = 50, N = 641$

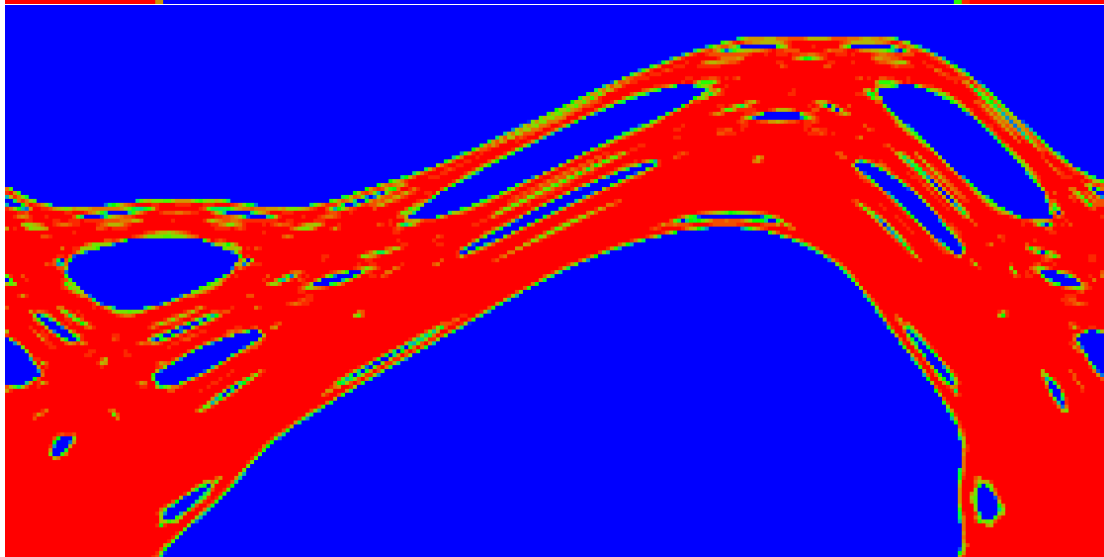


$\lambda_0 = 1, M = 70, N = 1189$



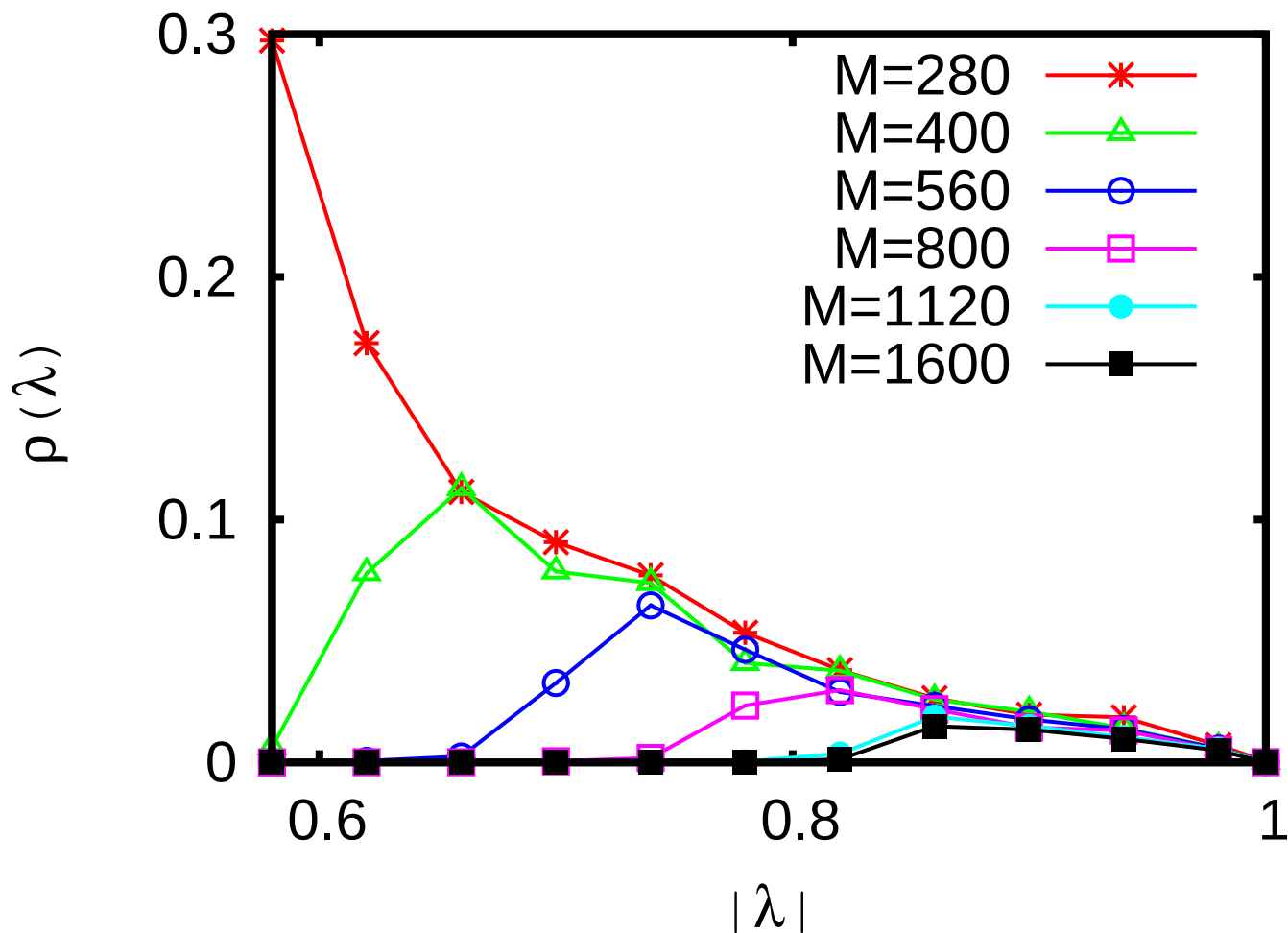


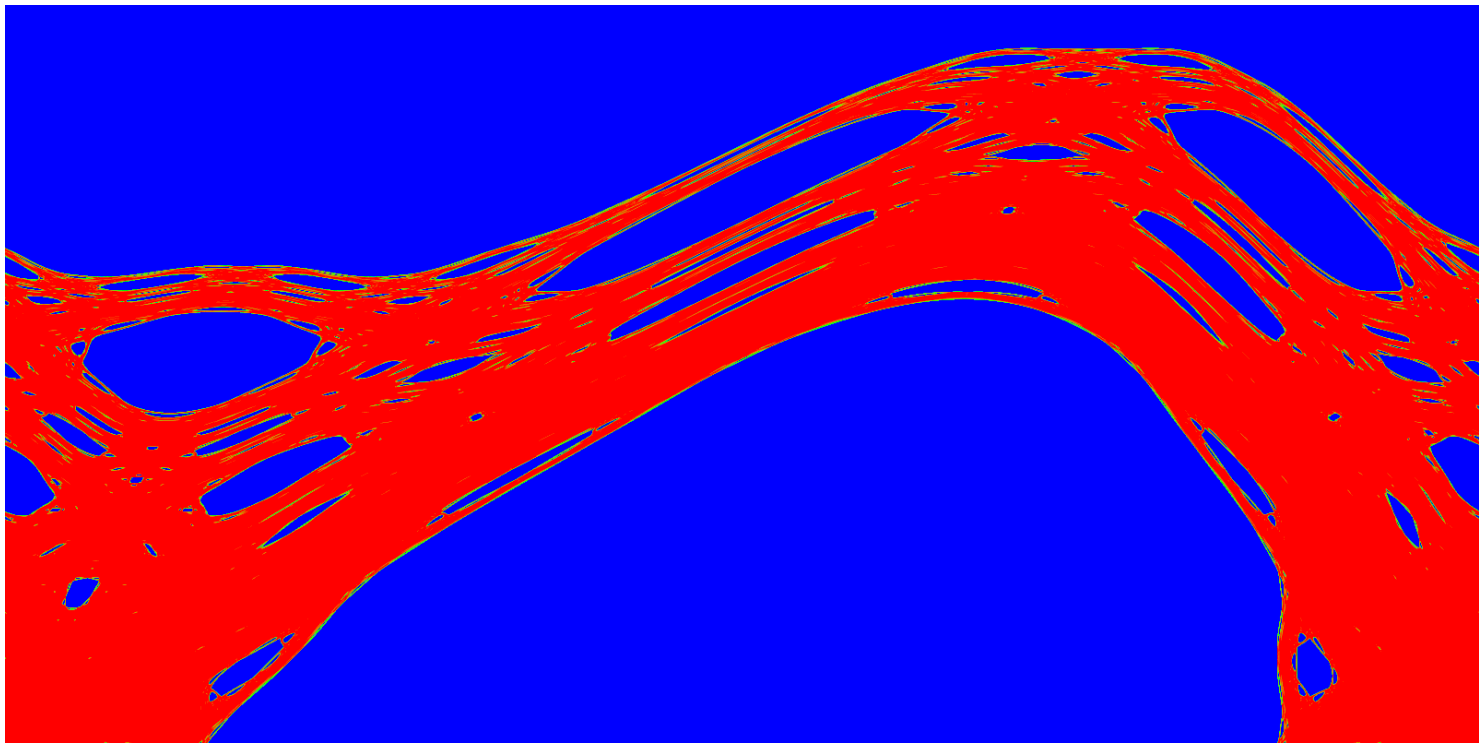
$\lambda_0 = 1$
 $M = 140$
 $N = 4417$



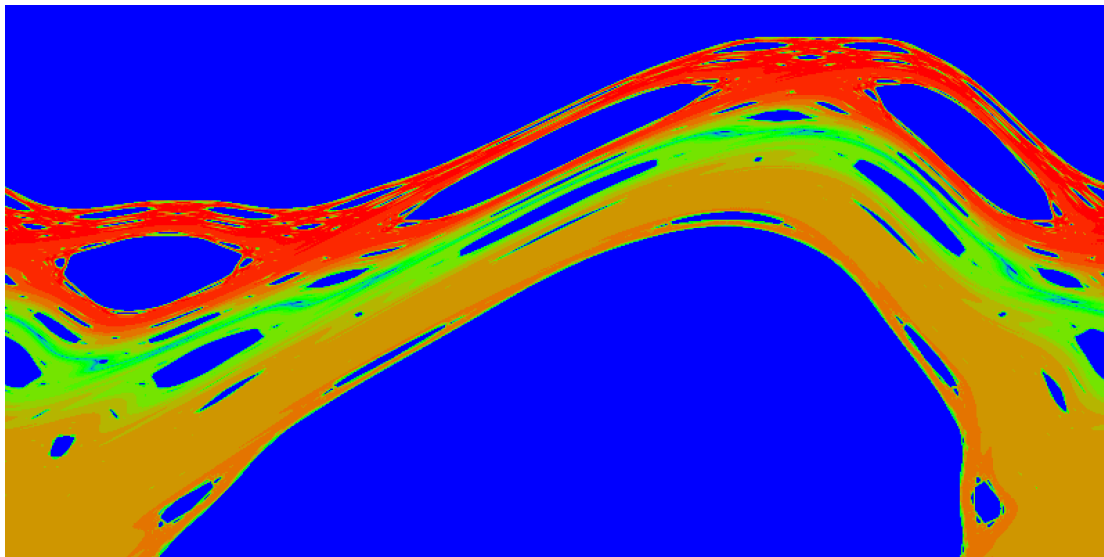
$\lambda_0 = 1$
 $M = 280$
 $N = 16609$

complex density of states:





$$\lambda_0 = 1, M = 1600, N = 494964, n_A = 3000$$



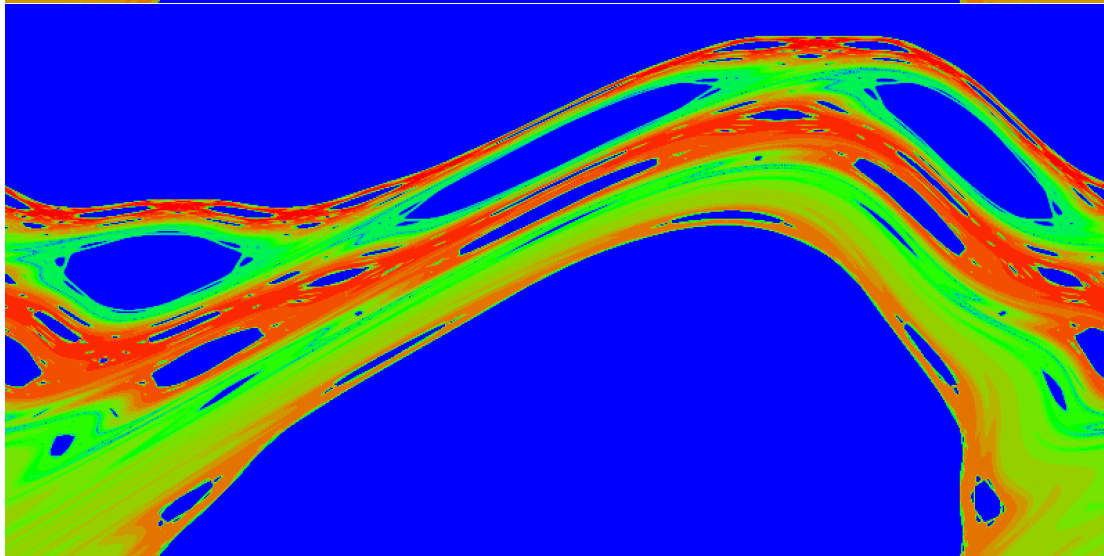
$$\lambda_1 =$$

$$0.99980431$$

$$M = 800$$

$$N = 127282$$

$$n_A = 2000$$



$$\lambda_2 =$$

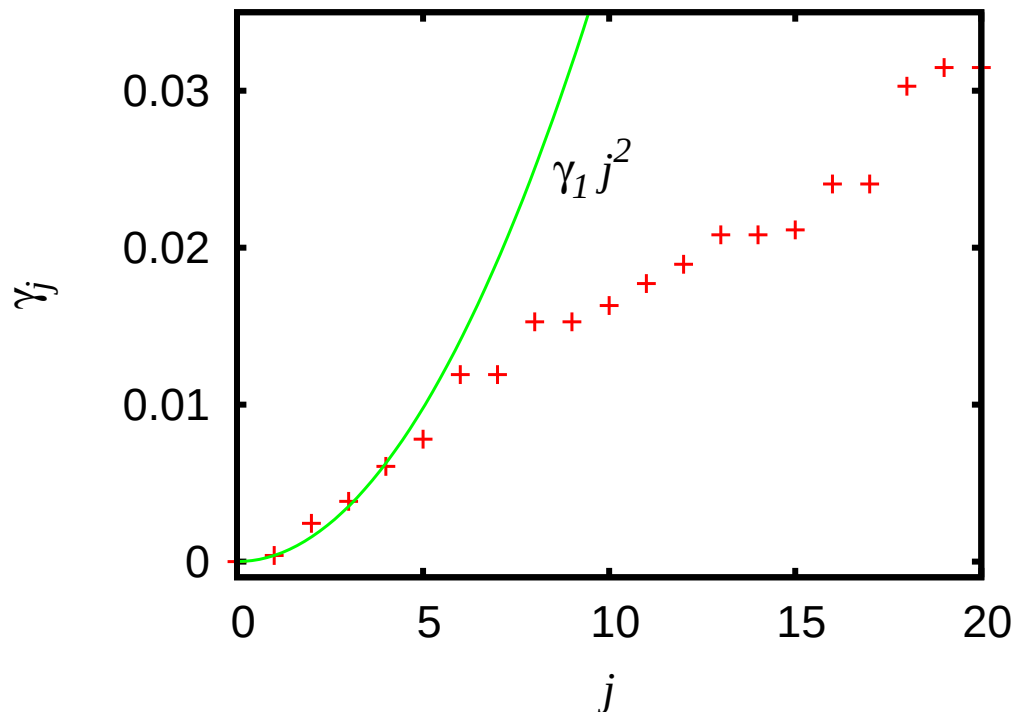
$$0.99878108$$

$$M = 800$$

$$N = 127282$$

$$n_A = 2000$$

Diffuson modes

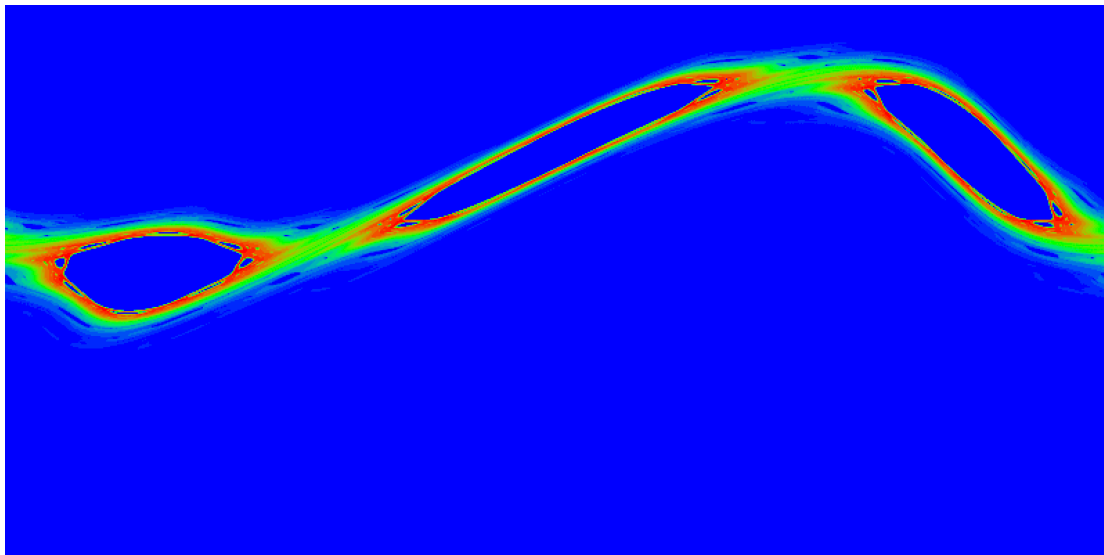


$$\gamma_j = -2 \ln(|\lambda_j|)$$

$$\gamma_j \approx \gamma_1 j^2$$

for $j \leq 5$.

What about eigenvectors for complex or real negative λ_j ?

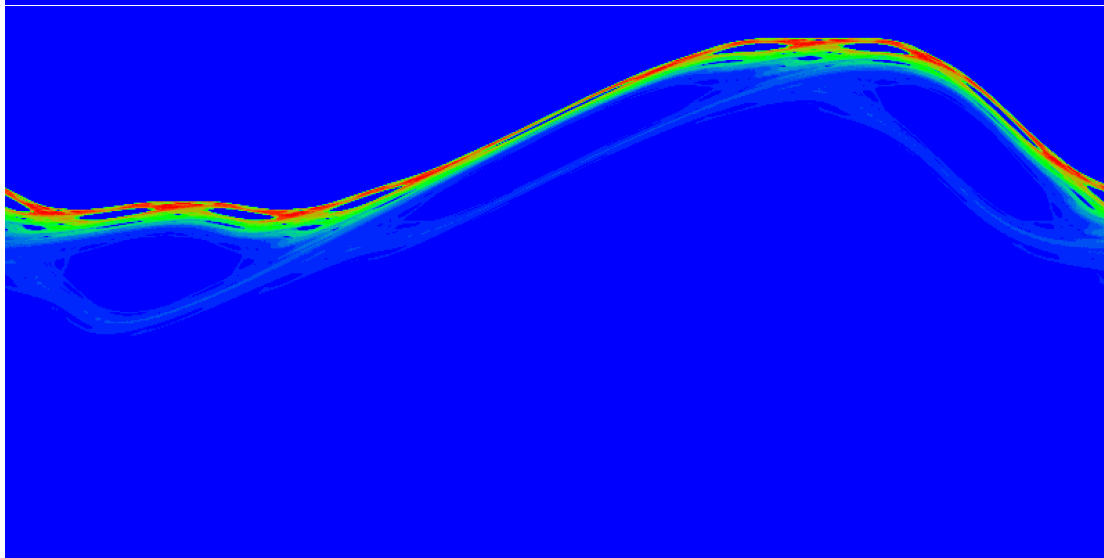


$$\begin{aligned}\lambda_6 &= \\ &-0.49699831 \\ &+i\,0.86089756 \\ &\approx |\lambda_6| e^{i\,2\pi/3}\end{aligned}$$

$$M = 800$$

$$N = 127282$$

$$n_A = 2000$$

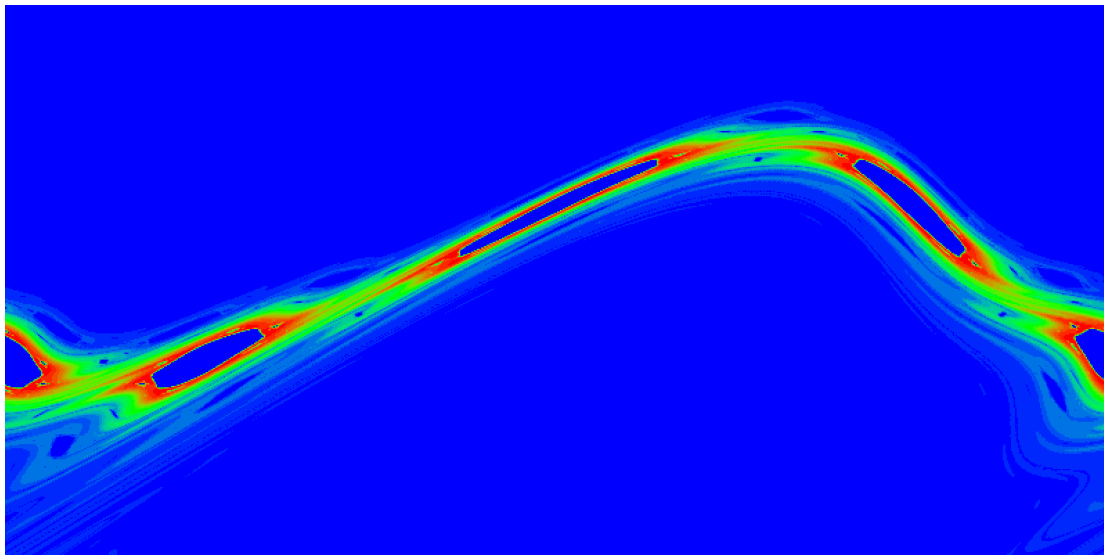


$$\begin{aligned}\lambda_{19} &= \\ &-0.71213331 \\ &+i\,0.67961609 \\ &\approx |\lambda_{19}| e^{i\,2\pi(3/8)}\end{aligned}$$

$$M = 800$$

$$N = 127282$$

$$n_A = 2000$$



$$\lambda_8 =$$

$$0.00024596$$

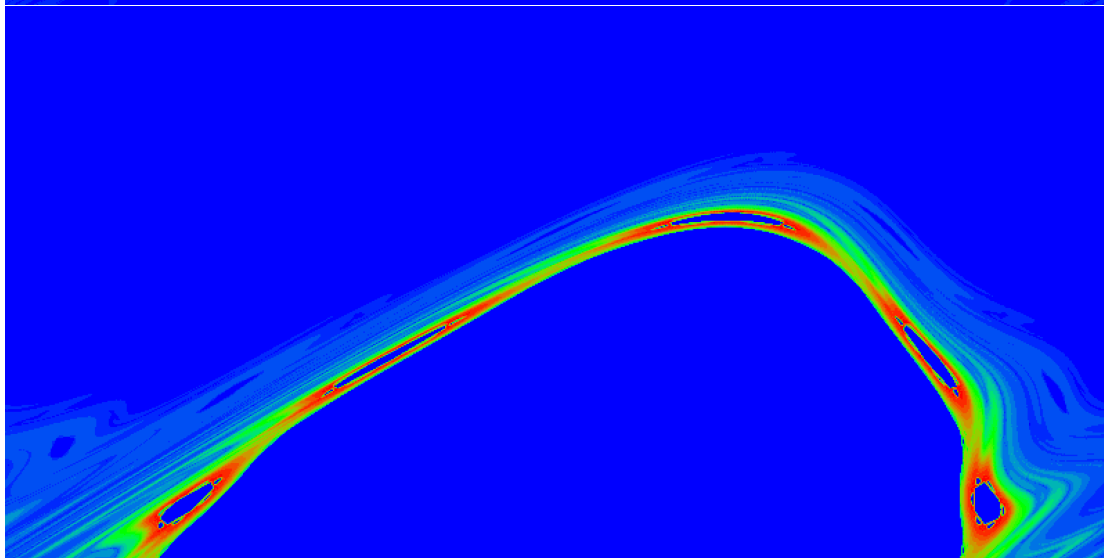
$$+i\,0.99239222$$

$$\approx |\lambda_8| e^{i\,2\pi/4}$$

$$M = 800$$

$$N = 127282$$

$$n_A = 2000$$



$$\lambda_{13} =$$

$$0.30580631$$

$$+i\,0.94120900$$

$$\approx |\lambda_{13}| e^{i\,2\pi/5}$$

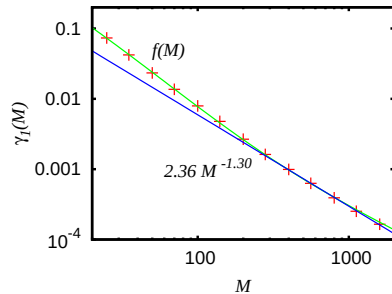
$$M = 800$$

$$N = 127282$$

$$n_A = 2000$$

Extrapolation of eigenvalues

$\gamma_1(M)$ in the limit $M \rightarrow \infty$:



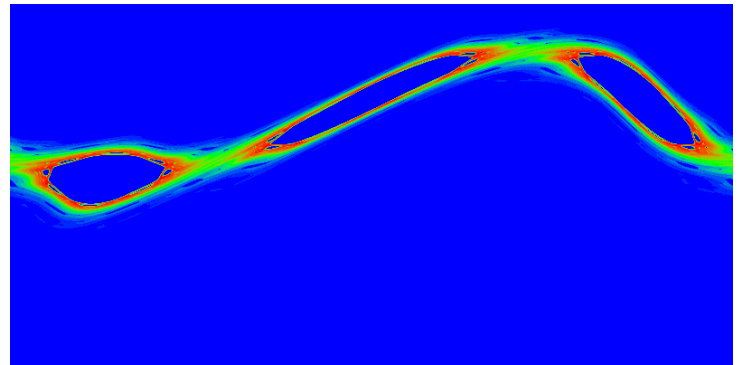
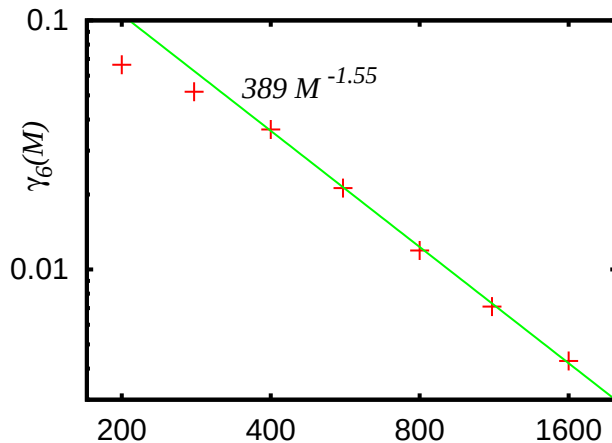
$$f(M) = \frac{D}{M} \frac{1 + \frac{C}{M}}{1 + \frac{B}{M}}$$

$$D = 0.245$$

$$B = 13.1$$

$$C = 258$$

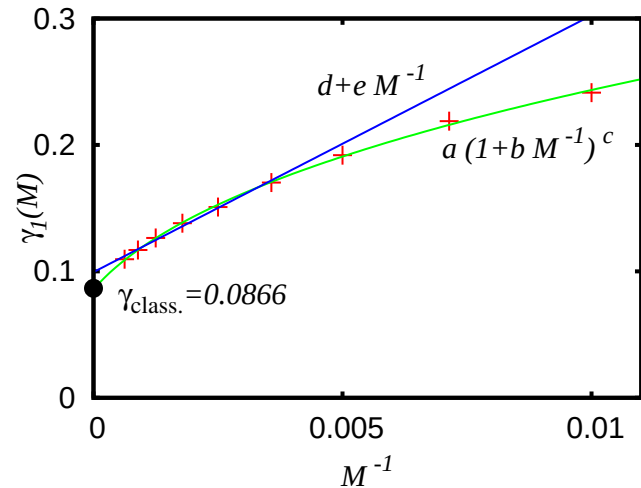
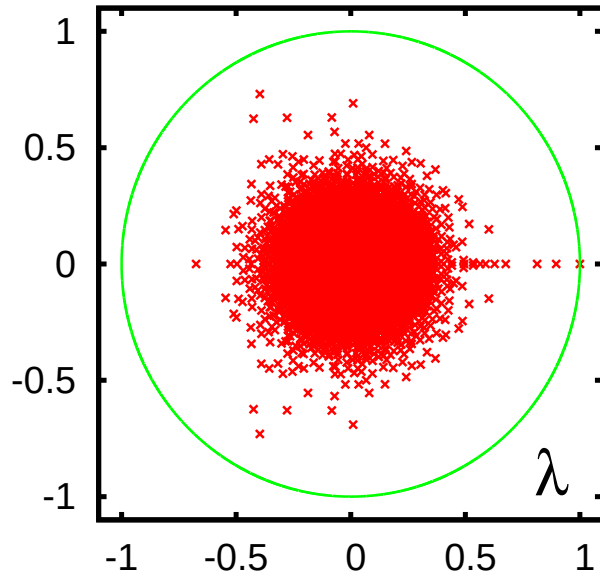
$\gamma_6(M)$ in the limit $M \rightarrow \infty$:



$$\gamma_6(M) \approx 389 M^{-1.55} \quad \text{for } M \geq 400.$$

Strong chaos

at $k = 7$, $M = 140$, $t = 10^{11}$ and $N = 9800$



$$a \approx 0.0857 \pm 0.0036, d \approx 0.0994$$

$$\lim_{M \rightarrow \infty} \gamma_1(M) > 0$$



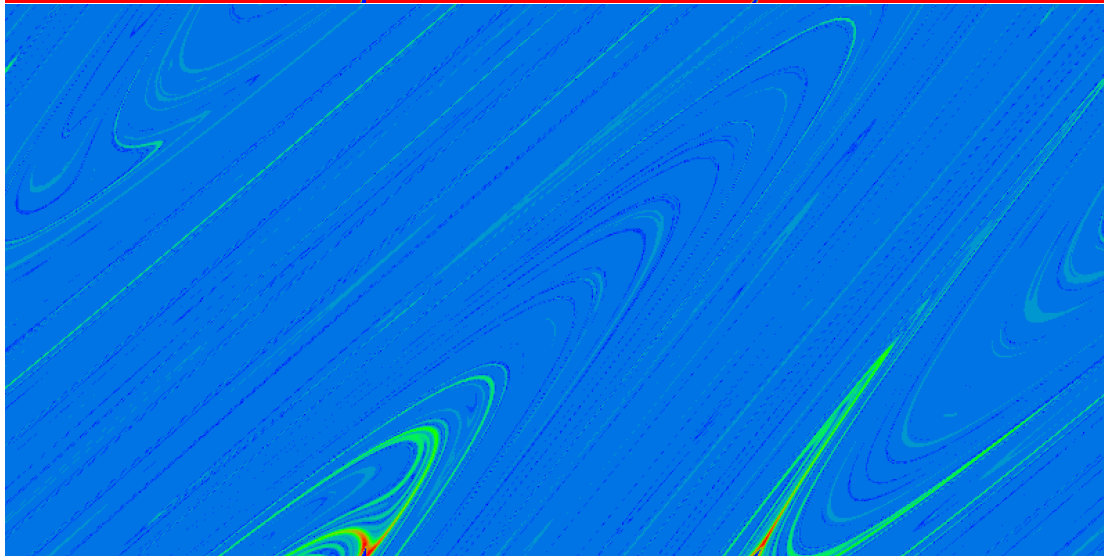
$$\lambda_0 = 1$$

$$k = 7$$

$$M = 800$$

$$N = 319978$$

$$n_A = 1500$$



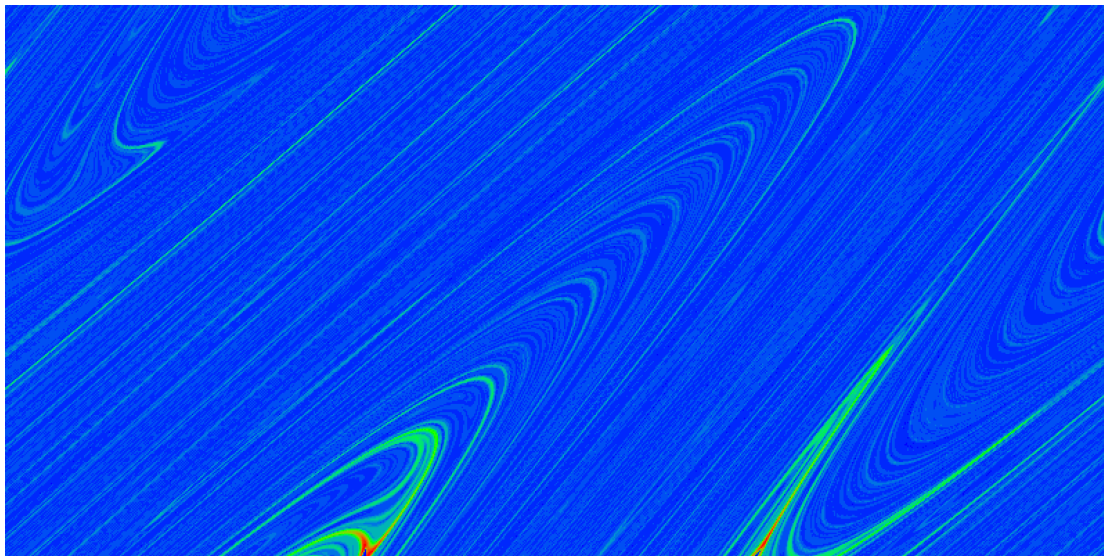
$$\lambda_1 = 0.93874817$$

$$k = 7$$

$$M = 800$$

$$N = 319978$$

$$n_A = 1500$$



$$\lambda_2 =$$

$$-0.49264273$$

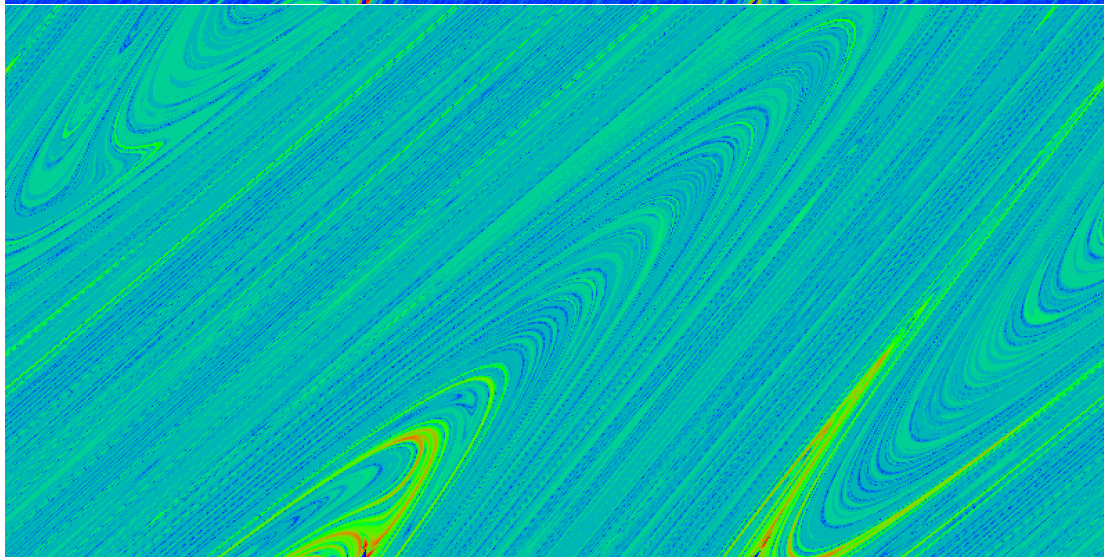
$$+i0.78912368$$

$$k = 7$$

$$M = 800$$

$$N = 319978$$

$$n_A = 1500$$



$$\lambda_8 =$$

$$0.87305253$$

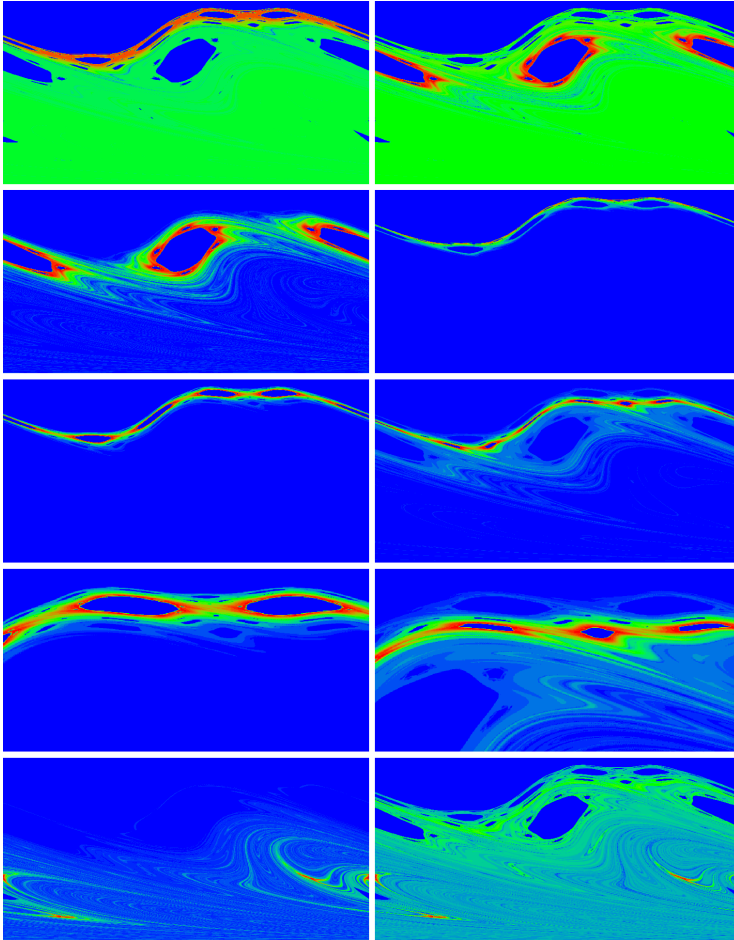
$$k = 7$$

$$M = 800$$

$$N = 319978$$

$$n_A = 1500$$

Separatrix map

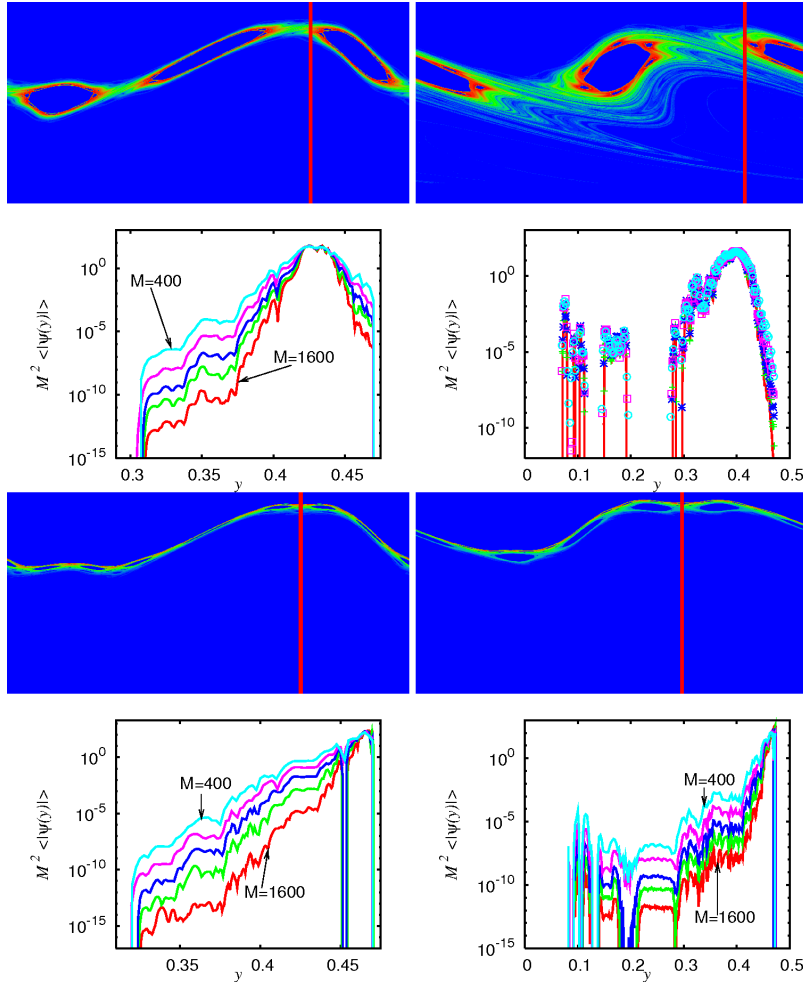


$$\bar{p} = p + \sin(2\pi x)$$

$$\bar{x} = x + \frac{\Lambda}{2\pi} \ln(|\bar{p}|) \pmod{1}$$

$$\Lambda = \Lambda_c = 3.1819316$$

Phase space localization



References

1. D. L. Shepelyansky ***Fractal Weyl law for quantum fractal eigenstates***, Phys. Rev. E **77**, p.015202(R) (2008).
2. L. Ermann and D. L. Shepelyansky, ***Ulam method and fractal Weyl law for Perron-Frobenius operators***, Eur. Phys. J. B **75**, 299 (2010).
3. K. M. Frahm and D. L. Shepelyansky, ***Ulam method for the Chirikov standard map***, Eur. Phys. J. B **76**, 57 (2010).
4. L. Ermann, K. M. Frahm and D. L. Shepelyansky, **Spectral properties of Google matrix of Wikipedia and other networks**, Eur. Phys. J. B **86**, 193 (2013).
5. K. M. Frahm and D. L. Shepelyansky, **Poincaré recurrences and Ulam method for the Chirikov standard map**, Eur. Phys. J. B **86**, 322 (2013).